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ZÁPADOČESKÉ UNIVERZITY
V PLZNI

KATEDRA
MATEMATIKY

KMA/APG1 Applications of Geometry 1

Differential Geometry of Curves

Differential Geometry – Introduction

What differential geometry is about

- Differential geometry studies curves and surfaces as smooth objects in space.
- Using tools from calculus and linear algebra, it investigates their local and global geometry.
- Throughout the course we will work in Euclidean space

$$\mathbb{E}_n = (\mathbb{R}^n,\ \mathbb{R}^n,\ +)$$

  with the standard inner product.

Why it matters

- It teaches us how to measure shape: lengths, angles, areas, and curvatures.
- It distinguishes intrinsic properties (measurable directly "on the surface": length, angles, area) from extrinsic ones (describing how the surface is bent in space).
- It is a natural language for geometric modeling, computer graphics, physics, and robotics.

Main topics

- For curves we introduce the tangent and normal, curvature, and in space also torsion (the Frenet frame).
- For surfaces we introduce the tangent plane and normal, the first and second fundamental forms, and normal curvature.
- From normal curvature we derive the principal curvatures, Gaussian and mean curvature, and the basic types of surface points.

Outline

Parametrized Curve

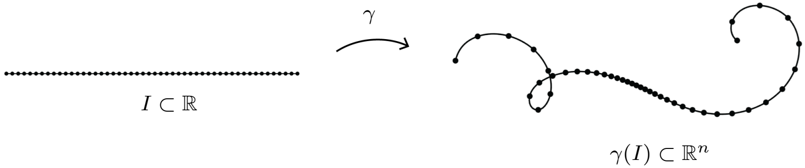
Definition (Parametrized curve)

A parametrized (parametric) curve in $\mathbb{R}^n$ is a smooth map $\gamma\colon I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval.

- By an interval we mean a nonempty connected subset of $\mathbb{R}$. Every interval has one of the following forms:

$$(a,b),\ [a,b],\ (a,b],\ [a,b),\ (-\infty,b),\ (-\infty,b],\ (a,\infty),\ [a,\infty),\ (-\infty,\infty).$$

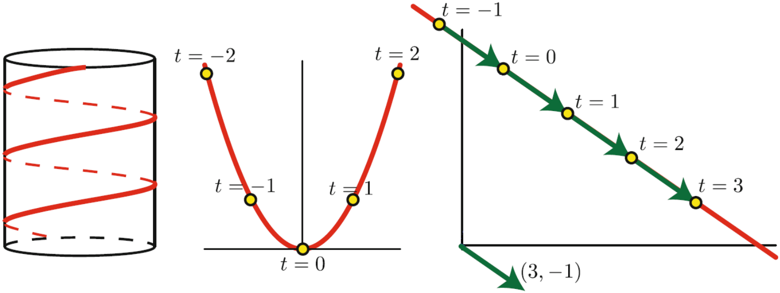
- A parametrized curve is therefore a specific way of traversing the points of the curve – including information about direction, speed, acceleration, etc.



Examples of curve parametrizations

Example (Examples of parametrizations)

Examples of specific parametric curves: a helix, a parabola, a line.



$\gamma(t) = [\cos t, \sin t, t]$

$\gamma(t) = [t, t^2]$

$\gamma(t) = [2, 4] + t(3, -1)$

Derivative of a curve

Definition (Derivative of a curve)

Let $\gamma\colon I \to \mathbb{R}^n$ be a parametric curve with components

$$\gamma(t) = [x_1(t), x_2(t), \ldots, x_n(t)].$$

Its derivative $\gamma'\colon I \to \mathbb{R}^n$ is defined by

$$\gamma'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t)).$$

Higher-order derivatives are defined analogously.

- We differentiate a parametric curve componentwise.
- For example, a space curve $\gamma(t) = [x(t), y(t), z(t)]$ has first derivative

  $$\gamma'(t) = (x'(t), y'(t), z'(t)),$$

  second derivative

  $$\gamma''(t) = (x''(t), y''(t), z''(t)),$$

  and so on.

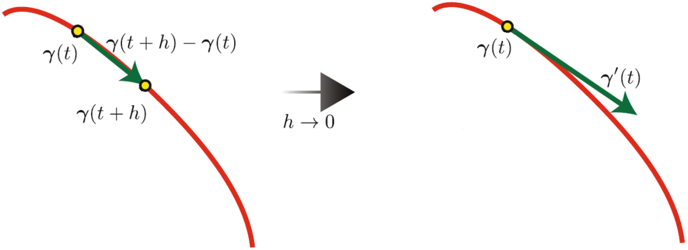
Geometric meaning of the derivative

Proposition (Geometric meaning of the derivative)

The derivative of a parametric curve $\gamma : I \to \mathbb{R}^n$ at time $t \in I$ is given by

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

Proof: This follows directly from the definition of the derivative and the fact that we differentiate componentwise. □



Smoothness of curves

Definition (A C^k curve)

A parametrized curve

$$\gamma : I \to \mathbb{R}^n$$

is of class C^k if all of its component functions have continuous derivatives up to order k (inclusive).

- Geometric constructions use derivatives of the curve of various orders.
- In particular:
  - defining the tangent requires a curve of the class at least C^1,
  - defining curvature requires a curve of the class at least C^2,
  - defining torsion requires a curve of the class at least C^3,
- To avoid technicalities at every step, we will not explicitly state the required smoothness class each time.
- Throughout the course we will implicitly assume that curves have as many continuous derivatives as needed for the construction at hand.
- Common functions used in mathematics and applications (polynomials, trigonometric functions, the exponential function, etc.) satisfy these smoothness assumptions automatically.

Curve

Definition (A curve as the image of a parametrization)

A curve (the trace of a parametrization) is the image of the map γ, i.e.,

$$C = \gamma(I) = \{\gamma(t) \in \mathbb{R}^n \mid t \in I\}.$$

- This set does not preserve information about direction or speed – it is only the geometric shape.
- When there is no risk of confusion, we will also use the term curve for the map γ itself (i.e., the parametrized curve).

Example (Different parametrizations of the same curve)

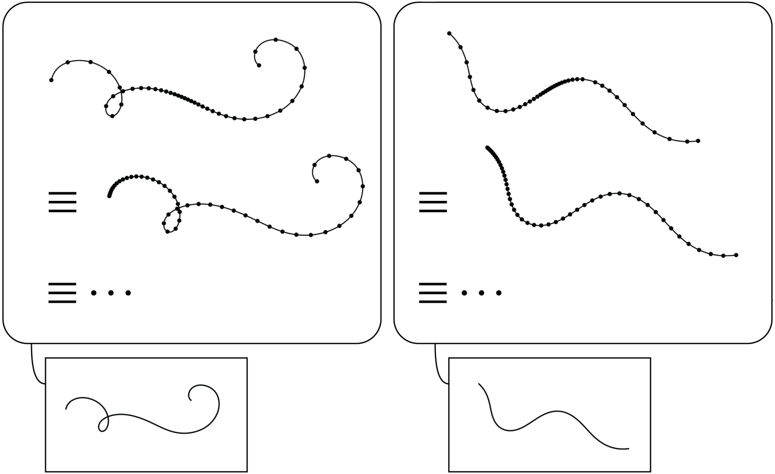
The parametrizations

$$\begin{aligned} \gamma_1(t) &= [\cos t, \sin t], & t &\in [0, 2\pi), \\ \gamma_2(s) &= [\cos(2s), \sin(2s)], & s &\in [0, \pi), \end{aligned}$$

have the same image (the unit circle in $\mathbb{R}^2$) but define different parametrized curves (different traversal and speed).

Parametrization vs. curve

- The function γ does not only describe the trajectory of a point; it also contains information about its kinematics, for instance velocity and acceleration.



Reparametrization

Definition (Reparametrization of a curve)

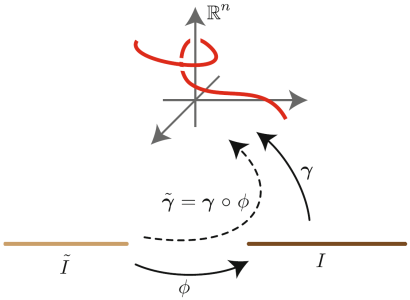
Let $\gamma : I \to \mathbb{R}^n$ be a parametrized curve. A reparametrization of γ is the composed curve

$$\tilde{\gamma}(t) = \gamma(\phi(t)),$$

where $\phi : \tilde{I} \to I$ is a smooth (class C^1) map between open intervals satisfying:

- ϕ is a bijection,
- $\phi'(t) \neq 0$ for all $t \in \tilde{I}$.

- When working with closed intervals, we interpret this as a restriction to their interiors.



Reparametrization

Example (An example of a reparametrization)

Consider two regular planar curves:

$$\gamma(t) = \left[t, t^2\right], \quad t \in [-2, 2],$$

$$\tilde{\gamma}(t) = \left[2t, (2t)^2\right], \quad t \in [-1, 1].$$

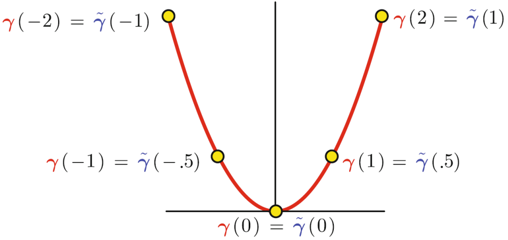
Both have the same trace – a part of a parabola. Define

$$\phi : [-1, 1] \to [-2, 2], \quad \phi(t) = 2t.$$

Then

$$\tilde{\gamma}(t) = \gamma(\phi(t)) = (\gamma \circ \phi)(t).$$

Thus $\tilde{\gamma}$ is a reparametrization of γ.



Regular curve

Definition (Speed and regularity)

Let $\gamma : I \to \mathbb{R}^n$ be a parametrized curve, where $I \subseteq \mathbb{R}$ is an interval. The instantaneous speed at time $t \in I$ is the norm of the derivative:

$$v(t) = \|\gamma'(t)\|.$$

The curve is regular if it has nonzero speed at every point:

$$\|\gamma'(t)\| \neq 0 \quad \text{for all } t \in I.$$

Example (Example of a non-regular curve)

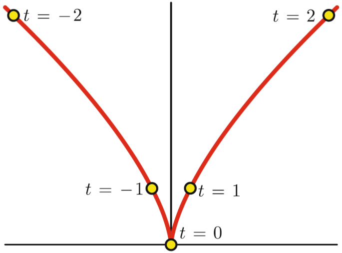
Consider the curve

$$\gamma(t) = \left[t^3, t^2\right], \quad t \in \mathbb{R}.$$

The curve has a cusp at the origin, since

$$\|\gamma'(0)\| = 0,$$

so γ is not regular at $t = 0$.



Closed and simple curves

Definition (Closed curve)

By a closed curve we mean a regular curve $\gamma : [a, b] \to \mathbb{R}^n$, such that

$$\gamma(a) = \gamma(b).$$

- In some situations we additionally require that all derivatives agree at the endpoints, i.e., $\gamma'(a) = \gamma'(b),\ \gamma''(a) = \gamma''(b),\ \ldots$ then we speak of a smoothly closed (equivalently: periodic) curve.

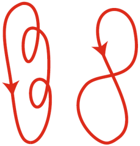
Definition (Simple curve)

A curve $\gamma : I \to \mathbb{R}^n$ is called simple if

$$\gamma(t_1) = \gamma(t_2) \ \Rightarrow\ t_1 = t_2 \quad \text{for all } t_1, t_2 \in I.$$

- If a simple curve is closed, we allow $\gamma(a) = \gamma(b)$.

closed & simple



not simple

Length of a curve

Definition (Arc length)

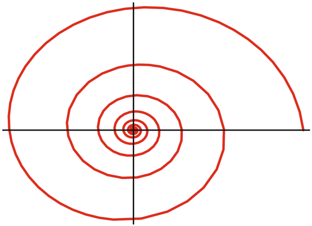
The arc length of a curve between times t_1 and t_2 (where $t_1, t_2 \in I$) is computed as the integral of the speed:

$$s(t_1, t_2) = \int_{t_1}^{t_2} \|\gamma'(t)\| \, dt.$$

Example (Logarithmic spiral)

$$\gamma(t) = c\big[e^{\lambda t}\cos t,\ e^{\lambda t}\sin t\big], \quad t \in \mathbb{R},\ c, \lambda \in \mathbb{R} \setminus \{0\}.$$

Show that for $\lambda < 0$ the curve γ has finite length on $[0, \infty)$, even though the spiral winds around the origin infinitely many times.



Length of a curve

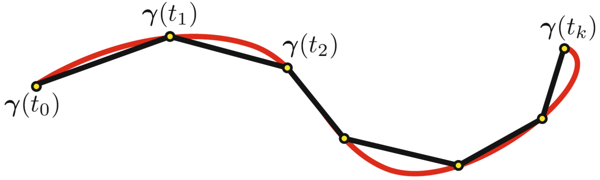
Proposition (Approximating length by a polygonal line)

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a regular curve. The arc length is given as the limit of the sum of lengths of a polygonal approximation:

$$\lim_{\delta \to 0} \sum_{i=0}^{k-1} \|\gamma(t_{i+1}) - \gamma(t_i)\| = \int_a^b \|\gamma'(t)\| \, dt,$$

where $a = t_0 < t_1 < \cdots < t_k = b$ is a partition of the interval and $\delta = \max_{0 \leq i < k}(t_{i+1} - t_i)$ is the norm of the partition (mesh size).

Proof: In the limit $\delta \to 0$, the polygonal length sum converges to the Riemann integral for arc length. □



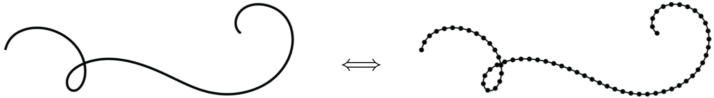
Arc-length parametrization

Definition (Arc-length parametrization)

A curve is parametrized by arc length if it has unit speed, i.e., if and only if

$$\|\gamma'(t)\| = 1 \quad \text{for all } t \in I.$$

- Arc-length parametrization is particularly convenient for computations and proofs.
- Every regular curve can in principle be reparametrized by arc length, although in practice this often leads to integrals that cannot be expressed in elementary functions.



Arc-length parametrization

Proposition (Existence of an arc-length parametrization)

Every regular curve can be reparametrized by arc length.

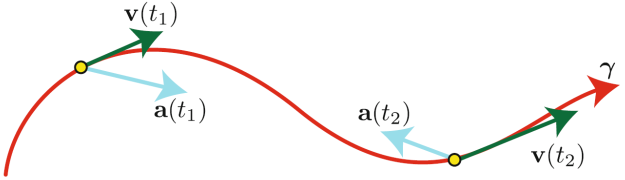
Proof:

- Since the curve is regular, we have $\|\gamma'(t)\| > 0$ for all $t \in I$.
- Fix $t_0 \in I$ and define the arc-length function
$$s(t) = \int_{t_0}^{t} \|\gamma'(u)\| \, du.$$
- The function $s(t)$ is smooth and strictly increasing (since $s'(t) = \|\gamma'(t)\| > 0$), hence it admits an inverse $t = \phi(s)$.
- Define a new parametrization
$$\tilde{\gamma}(s) = \gamma(\phi(s)).$$
- By the chain rule,
$$\tilde{\gamma}'(s) = \gamma'(\phi(s))\, \phi'(s).$$
- From the identity $s \circ \phi = \mathrm{id}$, i.e. $s(\phi(s)) = s$, differentiating yields
$$s'(\phi(s))\, \phi'(s) = 1 \quad \Rightarrow \quad \phi'(s) = \frac{1}{s'(\phi(s))} = \frac{1}{\|\gamma'(\phi(s))\|}.$$
- Hence
$$\|\tilde{\gamma}'(s)\| = \|\gamma'(\phi(s))\| \, |\phi'(s)| = \|\gamma'(\phi(s))\| \, \frac{1}{\|\gamma'(\phi(s))\|} = 1.$$

□

Velocity and acceleration vectors

- Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. We will use the following notation from physics:
  - velocity vector: $\mathbf{v}(t) = \gamma'(t)$
  - acceleration vector: $\mathbf{a}(t) = \gamma''(t)$
- The velocity vector $\mathbf{v}(t)$ encodes:
  - the direction of motion – it is tangent to the curve,
  - the speed of motion – its norm $v(t) = \|\mathbf{v}(t)\|$ is the instantaneous speed.

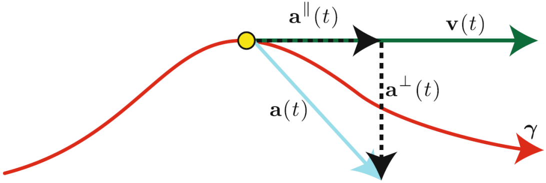


Acceleration as a "force" shaping the motion

- The acceleration $\mathbf{a}(t) = \gamma''(t)$ can be viewed as a force vector that "pulls" the object so that it follows the prescribed trajectory.
- In other words, $\mathbf{a}(t)$ describes how the velocity $\mathbf{v}(t)$ changes – that is, how the direction or the magnitude of the motion changes over time.
- It is natural to decompose the acceleration $\mathbf{a}(t)$ into two components:

$$\mathbf{a}(t) = \mathbf{a}^{\|}(t) + \mathbf{a}^{\perp}(t).$$

  - $\mathbf{a}^{\|}(t)$ is the component along the velocity $\mathbf{v}(t)$ – the tangential component,
  - $\mathbf{a}^{\perp}(t)$ is the component orthogonal to $\mathbf{v}(t)$ – the normal component.



Curvature of a curve

We introduce the curvature of a regular curve as a function

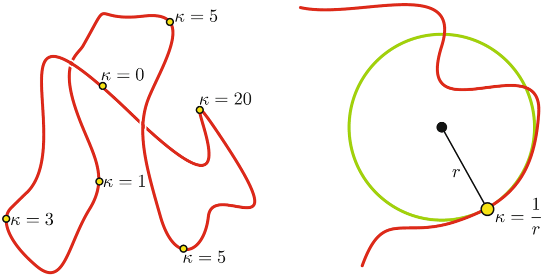
$$\kappa : I \to [0, \infty),$$

which at each $t \in I$ expresses how sharply the curve γ bends at the point $\gamma(t)$.

- The value $\kappa(t)$ is larger when the trajectory bends more strongly.
- If the curve looks locally like a straight line, then $\kappa(t) = 0$.
- For calibration we use a plane circle: If at $\gamma(t)$ the curve turns like a circle of radius r, then

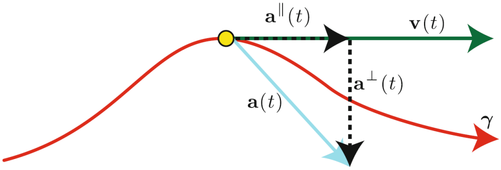
$$\kappa(t) = \frac{1}{r}.$$

  The smaller the circle, the larger the curvature.



Curvature of a curve

- How can we compute curvature?
- The acceleration vector $\mathbf{a}(t)$ describes how the velocity changes.
- Its normal component $\mathbf{a}^{\perp}(t)$ expresses how strongly the curve bends at a given point.



- However, the magnitude $\|\mathbf{a}^{\perp}(t)\|$ depends on the parametrization (the speed).

Definition (Definition of curvature)

Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. Its curvature is defined by

$$\kappa(t) = \frac{\|\mathbf{a}^{\perp}(t)\|}{\|\mathbf{v}(t)\|^2}.$$

Curvature of a curve

Proposition (Curvature is independent of parametrization)

Curvature is independent of parametrization.

Proof:

- Let $\tilde{\gamma}(t) = \gamma(\phi(t))$ be a reparametrization of the regular curve γ.
- Denote
$$\mathbf{v}(t) = \gamma'(t), \quad \mathbf{a}(t) = \gamma''(t), \quad \tilde{\mathbf{v}}(t) = \tilde{\gamma}'(t), \quad \tilde{\mathbf{a}}(t) = \tilde{\gamma}''(t).$$
- Then
$$\begin{aligned} \tilde{\mathbf{v}}(t) &= \mathbf{v}(\phi(t))\,\phi'(t) \\ \tilde{\mathbf{a}}(t) &= \mathbf{v}(\phi(t))\,\phi''(t) + \mathbf{a}(\phi(t))\,\phi'(t)^2 \\ \tilde{\mathbf{a}}^{\perp}(t) &= \mathbf{0} + \mathbf{a}^{\perp}(\phi(t))\,\phi'(t)^2. \end{aligned}$$
- Hence
$$\tilde{\kappa}(t) = \frac{\|\tilde{\mathbf{a}}^{\perp}(t)\|}{\|\tilde{\mathbf{v}}(t)\|^2} = \frac{\|\mathbf{a}^{\perp}(\phi(t))\|\,\phi'(t)^2}{\|\mathbf{v}(\phi(t))\|^2\,\phi'(t)^2} = \kappa(\phi(t)).$$

□

Curvature of a curve

Proposition (Curvature under arc-length parametrization)

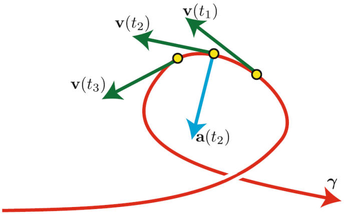
If γ is parametrized by arc length, then

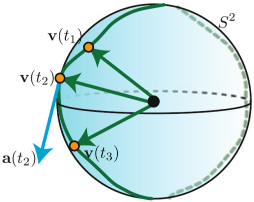
$$\kappa(t) = \|\mathbf{a}(t)\|.$$

Proof:

- Under arc-length parametrization ($\|\mathbf{v}(t)\| = 1$), the vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are orthogonal, so
$$\mathbf{a}^{\perp}(t) = \mathbf{a}(t).$$
- Therefore the curvature formula simplifies:
$$\kappa(t) = \frac{\|\mathbf{a}^{\perp}(t)\|}{\|\mathbf{v}(t)\|^2} = \frac{\|\mathbf{a}(t)\|}{1} = \|\mathbf{a}(t)\|.$$

□





Curvature of a circle

Example (Curvature of a circle)

- Consider a circle parametrized by

$$\gamma(t) = [r\cos t,\ r\sin t], \quad t \in [0, 2\pi).$$

- Velocity and acceleration:

$$\mathbf{v}(t) = \gamma'(t) = (-r\sin t,\ r\cos t), \qquad \mathbf{a}(t) = \gamma''(t) = (-r\cos t,\ -r\sin t).$$

- Note that $\mathbf{a}(t) = -\gamma(t)$: the acceleration points toward the center, which matches the physical intuition – motion along a circle requires a force directed inward.
- Moreover, $\mathbf{a}(t) \perp \mathbf{v}(t)$, hence

$$\mathbf{a}^{\perp}(t) = \mathbf{a}(t).$$

- Curvature:

$$\kappa(t) = \frac{\|\mathbf{a}^{\perp}(t)\|}{\|\mathbf{v}(t)\|^2} = \frac{\|\mathbf{a}(t)\|}{\|\mathbf{v}(t)\|^2} = \frac{r}{r^2} = \frac{1}{r}.$$

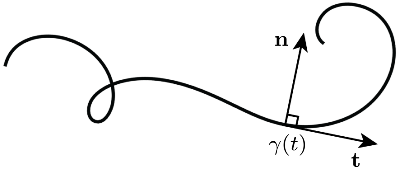
- Conclusion: A circle of radius r has constant curvature $\kappa = \frac{1}{r}$, in agreement with the calibration above.

Tangent and normal vectors

Definition (Tangent and normal vectors)

Let $\gamma : I \to \mathbb{R}^n$ be a regular curve and let $t \in I$ be such that $\|\mathbf{a}^\perp(t)\| \neq 0$. The unit tangent and unit normal vectors are defined by

$$\mathbf{t}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}, \qquad \mathbf{n}(t) = \frac{\mathbf{a}^\perp(t)}{\|\mathbf{a}^\perp(t)\|}.$$



- The tangent line to γ at the point $\gamma(t_0)$ is the line $\ell_{t_0} = \{\gamma(t_0) + \lambda\, \mathbf{t}(t_0) \mid \lambda \in \mathbb{R}\}$.

Alternative expression for curvature

Proposition (Curvature via the tangent vector)

Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. Then for all $t \in I$,

$$\kappa(t) = \frac{\|\mathbf{t}'(t)\|}{\|\mathbf{v}(t)\|}.$$

Proof:

- We know that

$$\mathbf{t}'(t) \perp \mathbf{t}(t).$$

- Differentiating $\mathbf{v}(t) = \|\mathbf{v}(t)\|\,\mathbf{t}(t)$ gives

$$\mathbf{v}'(t) = \mathbf{a}(t) = (\|\mathbf{v}(t)\|\,\mathbf{t}(t))' = \underbrace{\|\mathbf{v}(t)\|'\,\mathbf{t}(t)}_{\mathbf{a}^{\parallel}(t)} + \underbrace{\|\mathbf{v}(t)\|\,\mathbf{t}'(t)}_{\mathbf{a}^{\perp}(t)}.$$

- Therefore,

$$\kappa(t) = \frac{\|\mathbf{a}^{\perp}(t)\|}{\|\mathbf{v}(t)\|^2} = \frac{\|\mathbf{v}(t)\|\,\|\mathbf{t}'(t)\|}{\|\mathbf{v}(t)\|^2} = \frac{\|\mathbf{t}'(t)\|}{\|\mathbf{v}(t)\|}.$$

$\square$

- Thus, curvature measures the rate of change of tangent direction divided by speed (which makes it independent of parametrization).

Alternative expression for the normal vector

- From the proof of the previous proposition we know that

$$\mathbf{a}^{\perp} = \|\mathbf{v}\| \, \mathbf{t}'.$$

- The vectors $\mathbf{a}^{\perp}$ and $\mathbf{t}'$ point in the same direction and determine the unit normal.

Proposition (Expression for the normal vector)

Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. Then for all $t \in I$,

$$\mathbf{n} = \frac{\mathbf{a}^{\perp}}{\|\mathbf{a}^{\perp}\|} = \frac{\mathbf{t}'}{\|\mathbf{t}'\|}.$$

Alternative expression for curvature (cf. normal curvature of surfaces)

- We have:

$$\kappa = \frac{\|\mathbf{t}'\|}{\|\mathbf{v}\|}, \qquad \mathbf{n} = \frac{\mathbf{t}'}{\|\mathbf{t}'\|}.$$

- Hence

$$\mathbf{t}' = \|\mathbf{t}'\| \, \mathbf{n} = \kappa \, \|\mathbf{v}\| \, \mathbf{n}.$$

- Taking the scalar product of both sides with $\mathbf{n}$ gives:

$$\mathbf{t}' \cdot \mathbf{n} = \kappa \, \|\mathbf{v}\| \underbrace{\mathbf{n} \cdot \mathbf{n}}_{1} = \kappa \, \|\mathbf{v}\|.$$

- Differentiating the identity $\mathbf{t} \cdot \mathbf{n} = 0$ yields

$$\mathbf{t}' \cdot \mathbf{n} = -\mathbf{n}' \cdot \mathbf{t},$$

  and therefore:

Proposition (Another expression for curvature)

Let $\gamma : I \to \mathbb{R}^n$ be a regular curve. Then for all $t \in I$ we have:

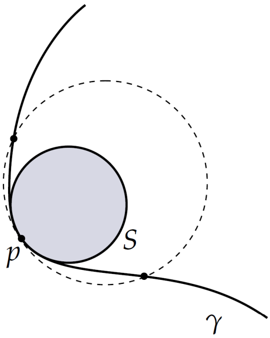
$$\kappa(t) = -\frac{\mathbf{n}'(t) \cdot \mathbf{t}(t)}{\|\mathbf{v}(t)\|}.$$

Osculating circle

- There is a unique circle passing through three (non-collinear) points $\gamma(t_0 - h)$, $\gamma(t_0)$, $\gamma(t_0 + h)$.
- As $h \to 0$, this circle converges to the osculating circle at the point $\gamma(t_0)$.

Definition (Osculating circle)

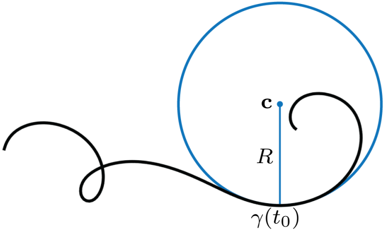
The osculating circle is the unique circle that at $\gamma(t_0)$ has the same tangent direction and the same curvature as the curve γ.



Osculating circle

- The osculating circle has the following center and radius (at points where $\kappa(t_0) \neq 0$):

$$\mathbf{c} = \gamma(t_0) + R\,\mathbf{n}(t_0), \qquad R = \frac{1}{\kappa(t_0)}.$$



- If $\kappa(t_0) = 0$, the osculating circle degenerates to the tangent line at $\gamma(t_0)$.
- If $\kappa(t_0) \neq 0$, the osculating circle lies in the osculating plane.

Definition (Osculating plane)

Assume $\kappa(t_0) \neq 0$. The plane spanned by $\mathbf{t}(t_0)$ and $\mathbf{n}(t_0)$ is called the osculating plane at time t_0:

$$\text{osculating plane} = \operatorname{span}\{\mathbf{t}(t_0), \mathbf{n}(t_0)\} = \operatorname{span}\{\mathbf{v}(t_0), \mathbf{a}(t_0)\}.$$

Outline

Signed curvature

- In the plane $\mathbb{R}^2$ we can naturally distinguish the direction of turning: clockwise vs. counterclockwise.
- Define the map $R_{90}(x, y) = (-y, x)$, which rotates a vector by 90° counterclockwise.
- For a plane curve $\gamma : I \to \mathbb{R}^2$ parametrized by arc length, the acceleration vector $\mathbf{a}(t)$ is parallel to $R_{90}(\mathbf{v}(t))$.

Definition (Signed curvature)

Let $\gamma : I \to \mathbb{R}^2$ be a plane curve parametrized by arc length. Then there exists a function $\kappa_s : I \to \mathbb{R}$, called the signed curvature, such that

$$\mathbf{a}(t) = \kappa_s(t)\, R_{90}(\mathbf{v}(t)).$$

Signed curvature – computation

Proposition (Formula for signed curvature)

Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve (not necessarily parametrized by arc length). Then

$$\kappa_s(t) = \frac{\mathbf{a}(t) \cdot R_{90}(\mathbf{v}(t))}{\|\mathbf{v}(t)\|^3}.$$

Proof:

- For an arc-length parametrization we have

$$\kappa_s(t) = \mathbf{a}(t) \cdot R_{90}(\mathbf{v}(t)).$$

- For a general parametrization,

$$\kappa_s(t) = \frac{\mathbf{a}(t) \cdot R_{90}\left(\frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}\right)}{\|\mathbf{v}(t)\|^2} = \frac{\mathbf{a}(t) \cdot R_{90}(\mathbf{v}(t))}{\|\mathbf{v}(t)\|^3}.$$

□

- For $\gamma(t) = [x(t), y(t)]$ we obtain

$$\kappa_s(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}.$$

Relation between signed and unsigned curvature

Lemma (Relation between κ_s and κ)

Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve. Then for every $t \in I$,

$$|\kappa_s(t)| = \kappa(t).$$

Proof:

- Choose an arc-length parametrization, so $\|\mathbf{v}(t)\| = 1$.
- By the definition of signed curvature,
$$\mathbf{a}(t) = \kappa_s(t)\, R_{90}(\mathbf{v}(t)),$$
hence
$$\|\mathbf{a}(t)\| = |\kappa_s(t)|.$$
- At the same time, for arc-length parametrization we have
$$\kappa(t) = \|\mathbf{a}(t)\|.$$

□

Area enclosed by a curve

Proposition (Green's theorem for area)

Let $\gamma : [a, b] \to \mathbb{R}^2$, where $\gamma(t) = [x(t), y(t)]$, be a simple, regular, oriented, and closed plane curve. Then the area enclosed by the curve is

$$A = \frac{1}{2} \int_a^b \big(x(t)y'(t) - y(t)x'(t) \big) \, dt.$$

Equivalently,

$$A = \int_a^b x(t)y'(t) \, dt = -\int_a^b y(t)x'(t) \, dt.$$

- An oriented curve is a curve with a chosen traversal direction.
- The integral gives the signed area: it is positive for counterclockwise orientation and negative for clockwise orientation.

Example (Area of an ellipse)

Compute the area enclosed by the ellipse parametrized by

$$\gamma(t) = [a \cos t, \, b \sin t], \quad t \in [0, 2\pi].$$

Angle function

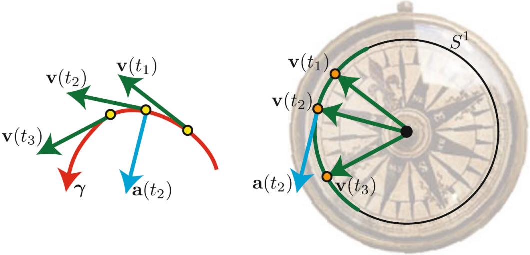
- For a plane curve $\gamma : I \to \mathbb{R}^2$ with unit speed, we have $\mathbf{v}(t) \in S^1$.
- The direction of motion can be described by an angle function $\theta(t)$ tracking the orientation of the velocity vector $\mathbf{v}(t)$.
- The signed curvature $\kappa_s(t)$ is the turning rate of the tangent direction.

Proposition (Existence of an angle function)

Let $\gamma : I \to \mathbb{R}^2$ be a plane curve parametrized by arc length. Then there exists a smooth angle function $\theta : I \to \mathbb{R}$ such that

$$\gamma'(t) = \mathbf{v}(t) = (\cos\theta(t), \sin\theta(t)).$$

It is uniquely determined up to adding an integer multiple of 2π.



Rotation index of a plane curve

- The rotation index is the total number of counterclockwise turns of the tangent direction along the curve.

Definition (Rotation index)

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a regular closed plane curve (parametrized by arc length). Choose a continuous angle function $\theta : [a, b] \to \mathbb{R}$ such that

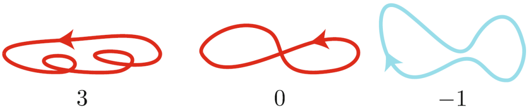
$$\mathbf{v}(t) = \gamma'(t) = (\cos\theta(t), \sin\theta(t)).$$

The rotation index of γ is defined by

$$\operatorname{ind}(\gamma) = \frac{1}{2\pi}\big(\theta(b) - \theta(a)\big).$$

- Since the curve is closed, $\theta(b) - \theta(a) \in 2\pi\mathbb{Z}$, hence $\operatorname{ind}(\gamma) \in \mathbb{Z}$.
- We also have the integral expression:

$$\operatorname{ind}(\gamma) = \frac{1}{2\pi}\int_a^b \kappa_s(t)\,\|\gamma'(t)\|\,dt.$$



Outline

Space curves

- In $\mathbb{R}^3$ we have the cross product, which allows us to express orthogonality and the area of a parallelogram via vectors.

Proposition (Curvature formula for a space curve)

Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve. Then

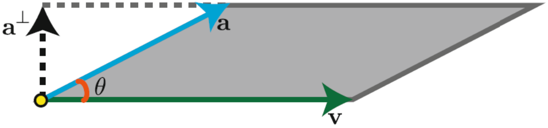
$$\kappa(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|^3}.$$

Proof:

- We have $\|\mathbf{a}^\perp(t)\| = \|\mathbf{a}(t)\| \sin\theta$, where θ is the angle between $\mathbf{v}(t)$ and $\mathbf{a}(t)$.
- Substituting into the definition of curvature gives

$$\kappa(t) = \frac{\|\mathbf{a}^\perp(t)\|}{\|\mathbf{v}(t)\|^2} = \frac{\|\mathbf{a}(t)\| \sin\theta}{\|\mathbf{v}(t)\|^2}.$$

- Using $\|\mathbf{v}(t) \times \mathbf{a}(t)\| = \|\mathbf{v}(t)\| \, \|\mathbf{a}(t)\| \sin\theta$ yields the stated formula. □



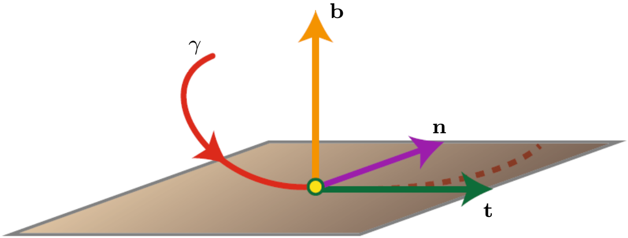
The Frenet frame in space

Definition (Frenet frame)

Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve and let $\kappa(t) \neq 0$ at $t \in I$. The Frenet frame at time t is the orthonormal basis of $\mathbb{R}^3$ given by

$$\mathbf{t}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}, \qquad \mathbf{n}(t) = \frac{\mathbf{a}^{\perp}(t)}{\|\mathbf{a}^{\perp}(t)\|} = \frac{\mathbf{t}'(t)}{\|\mathbf{t}'(t)\|}, \qquad \mathbf{b}(t) = \mathbf{t}(t) \times \mathbf{n}(t).$$

- $\mathbf{t}(t)$ – the unit tangent vector,
- $\mathbf{n}(t)$ – the unit normal vector,
- $\mathbf{b}(t)$ – the unit binormal vector.



Torsion

- Torsion measures how fast the osculating plane tilts along the curve γ.
- A natural candidate is $\|\mathbf{b}'(t)\|$, the magnitude of the derivative of the binormal, but this quantity:
  - depends on the parametrization,
  - does not encode the direction of change (it has no sign).
- From $\mathbf{b} \cdot \mathbf{b} = 1$ and $\mathbf{b} \cdot \mathbf{t} = 0$, differentiation gives $\mathbf{b}' \cdot \mathbf{b} = 0$ and $\mathbf{b}' \cdot \mathbf{t} = 0$, so $\mathbf{b}' \perp \mathbf{b}$ and $\mathbf{t}$.
- Hence $\mathbf{b}'(t)$ is parallel to $\mathbf{n}(t)$.
- Therefore we take the oriented rate of change in the direction of $\mathbf{n}(t)$ and divide by speed.

Definition (Torsion of a space curve)

Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve and let $\kappa(t) \neq 0$. The torsion of γ at $t \in I$ is defined by

$$\tau(t) = -\frac{\mathbf{b}'(t) \cdot \mathbf{n}(t)}{\|\mathbf{v}(t)\|}.$$

- From the parallelism of $\mathbf{b}'(t)$ and $\mathbf{n}(t)$ it follows that $\mathbf{b}'(t) = \|\mathbf{v}(t)\|\, \tau(t)\, \mathbf{n}(t)$ (see the Frenet formulas).

Invariance of torsion under reparametrization

Proposition (Torsion is invariant under orientation-preserving reparametrizations)

The torsion $\tau(t)$ is independent of orientation-preserving reparametrizations.

Proof:

- Let $\tilde{\gamma} = \gamma \circ \phi$ be a reparametrization. Quantities associated with $\tilde{\gamma}$ are marked with a tilde.
- If $\phi'(t) > 0$ (orientation preserved), then the Frenet frame transforms as

$$\tilde{\mathbf{t}} = \mathbf{t} \circ \phi, \qquad \tilde{\mathbf{n}} = \mathbf{n} \circ \phi, \qquad \tilde{\mathbf{b}} = \mathbf{b} \circ \phi.$$

- Then

$$\tilde{\tau}(t) = -\frac{\tilde{\mathbf{b}}'(t) \cdot \tilde{\mathbf{n}}(t)}{\|\tilde{\mathbf{v}}(t)\|} = -\frac{\phi'(t)\,\mathbf{b}'(\phi(t)) \cdot \mathbf{n}(\phi(t))}{|\phi'(t)|\,\|\mathbf{v}(\phi(t))\|} = \tau(\phi(t)).$$

- If $\phi'(t) < 0$ (orientation reversed), then

$$\tilde{\mathbf{t}} = -\mathbf{t} \circ \phi, \qquad \tilde{\mathbf{n}} = \mathbf{n} \circ \phi, \qquad \tilde{\mathbf{b}} = -\mathbf{b} \circ \phi,$$

  and an extra minus sign appears in the torsion, so

$$\tilde{\tau}(t) = -\tau(\phi(t)).$$

□

A torsion formula via the scalar triple product

Proposition (Torsion formula)

Let $\gamma : I \to \mathbb{R}^3$ be a three-times differentiable regular curve with $\kappa(t) \neq 0$. Then the torsion $\tau(t)$ is given by

$$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

Proof: Let $\mathbf{v} = \gamma'$, $\mathbf{a} = \gamma''$, $\mathbf{k} = \gamma'''$. Then:

- Frenet frame:

$$\mathbf{t} = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad \mathbf{n} = \frac{\mathbf{a}^\perp}{\|\mathbf{a}^\perp\|}, \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}.$$

- By definition,

$$\tau = -\frac{\mathbf{b}' \cdot \mathbf{n}}{\|\mathbf{v}\|}.$$

- One can show that

$$\mathbf{b} = \frac{\mathbf{v} \times \mathbf{a}}{\|\mathbf{v} \times \mathbf{a}\|}, \qquad \mathbf{b}' \cdot \mathbf{n} = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{k}}{\|\mathbf{v} \times \mathbf{a}\|^2}.$$

- Substituting yields

$$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

□

Torsion of a planar curve

Example (Torsion of a planar curve)

Let $\gamma : I \to \mathbb{R}^3$ be a space curve lying in a plane.

- Without loss of generality we may take this plane to be the xy-plane, i.e.,

$$\gamma(t) = [x(t), y(t), 0].$$

- At points where $\kappa(t) = 0$, the vectors $\mathbf{n}(t)$, $\mathbf{b}(t)$ and the torsion $\tau(t)$ are not defined.
- If $\kappa(t) \neq 0$, then $\mathbf{t}(t)$ and $\mathbf{n}(t)$ lie in the xy-plane, and their cross product is

$$\mathbf{b}(t) = \mathbf{t}(t) \times \mathbf{n}(t) = (0, 0, \pm 1),$$

  where the sign depends on the curve orientation (clockwise vs. counterclockwise).
- Since $\mathbf{b}(t)$ is constant on each interval, we have

$$\tau(t) = 0 \quad \text{at every point where it is defined.}$$

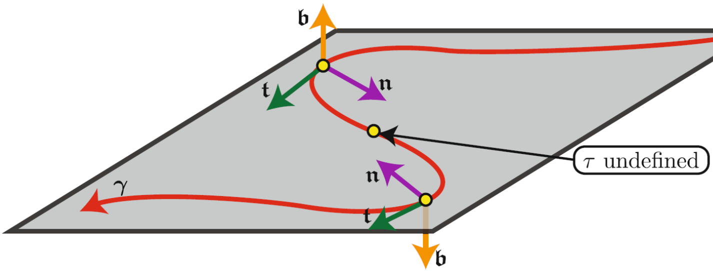
Curves with zero torsion

- Zero torsion means that the osculating plane does not change – so the curve stays in a single plane. We already saw this for planar curves.

Proposition (Characterization of zero torsion)

Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve and assume $\kappa(t) \neq 0$ for all $t \in I$. Then

$$\tau(t) = 0 \text{ for all } t \in I \quad \Longleftrightarrow \quad \text{the image of the curve lies in a plane.}$$



Frenet–Serret formulas

Proposition (Frenet–Serret formulas)

Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve with $\kappa(t) \neq 0$. Then

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \|\mathbf{v}(t)\| \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}.$$

Proof:

- Let
$$\mathbf{Q}(t) = \begin{pmatrix} \mathbf{t}(t) & \mathbf{n}(t) & \mathbf{b}(t) \end{pmatrix}.$$
Since $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is an orthonormal basis, we have $\mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$.
- Differentiating yields
$$\mathbf{Q}'\mathbf{Q}^\top = -(\mathbf{Q}'\mathbf{Q}^\top)^\top.$$
- Thus $\mathbf{A}(t) = \mathbf{Q}'\mathbf{Q}^\top$ is skew-symmetric and has the form
$$\mathbf{A} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}.$$
- From $\mathbf{t}' = \|\mathbf{v}\|\kappa\,\mathbf{n}$ we get $\alpha = \|\mathbf{v}\|\kappa$ and $\beta = 0$.
- From $\mathbf{b}' = \|\mathbf{v}\|\tau\mathbf{n}$ we obtain $\gamma = \|\mathbf{v}\|\tau$.
- Since $\mathbf{Q}' = \mathbf{A}\mathbf{Q}$, the Frenet–Serret formulas follow. □

Curve invariants under isometries

- An isometry may translate, rotate, or reflect a curve, but it does not change its shape.

Proposition (Invariants under isometries)

For a regular curve, the following quantities are invariant under proper isometries:

- curvature κ in $\mathbb{R}^n$,
- torsion τ of a space curve,
- signed curvature κ_s of a plane curve.

Improper isometries (e.g. reflections) preserve curvature κ, but reverse the sign of torsion and signed curvature:

$$\tau \mapsto -\tau, \qquad \kappa_s \mapsto -\kappa_s.$$

Fundamental theorems of plane and space curves

Theorem (Fundamental theorems of plane and space curves)

- Plane curves
  - Let $I \subset \mathbb{R}$ be an interval and let $\kappa_s : I \to \mathbb{R}$ be a smooth function. Then there exists an arc-length parametrized curve $\gamma : I \to \mathbb{R}^2$ whose signed curvature equals κ_s.
  - If $\gamma, \hat{\gamma} : I \to \mathbb{R}^2$ are two such curves, then there exists a proper isometry f of $\mathbb{R}^2$ such that
$$\hat{\gamma} = f \circ \gamma.$$
- Space curves
  - Let $I \subset \mathbb{R}$ be an interval and let $\kappa, \tau : I \to \mathbb{R}$ be smooth functions, with $\kappa(t) > 0$ for all $t \in I$. Then there exists an arc-length parametrized curve $\gamma : I \to \mathbb{R}^3$ whose curvature is κ and torsion is τ.
  - If $\gamma, \hat{\gamma} : I \to \mathbb{R}^3$ are two such curves, then there exists a proper isometry f of $\mathbb{R}^3$ such that
$$\hat{\gamma} = f \circ \gamma.$$

- A plane curve is uniquely determined (up to proper isometries) by its signed curvature.
- A space curve is uniquely determined (up to proper isometries) by its curvature and torsion.

Constructing a plane curve from signed curvature

- Let $I \subset \mathbb{R}$ and let $\kappa_s : I \to \mathbb{R}$ be a smooth function.
- Choose $t_0 \in I$ and, for simplicity, initial conditions
$$\gamma(t_0) = [0,0], \qquad \mathbf{v}(t_0) = (1,0).$$
- Define the angle function
$$\theta(t) = \int_{t_0}^{t} \kappa_s(u)\, du + 0.$$
- Define the velocity
$$\mathbf{v}(t) = (\cos\theta(t), \sin\theta(t)).$$
Then $\|\mathbf{v}(t)\| = 1$ and $\theta'(t) = \kappa_s(t)$.
- Finally, define the curve
$$\gamma(t) = \int_{t_0}^{t} \mathbf{v}(u)\, du + [0,0].$$
- The resulting curve is arc-length parametrized, has signed curvature κ_s, and satisfies the chosen initial conditions.
- Changing the initial conditions $\gamma(t_0)$ and $\mathbf{v}(t_0)$ produces all curves with the same signed curvature.