FAKULTA APLIKOVANÝCH VĚD
ZÁPADOČESKÉ UNIVERZITY
V PLZNI

KATEDRA
MATEMATIKY

KMA/APG1 Applications of Geometry 1

Differential Geometry of Surfaces

Parametrized Surface

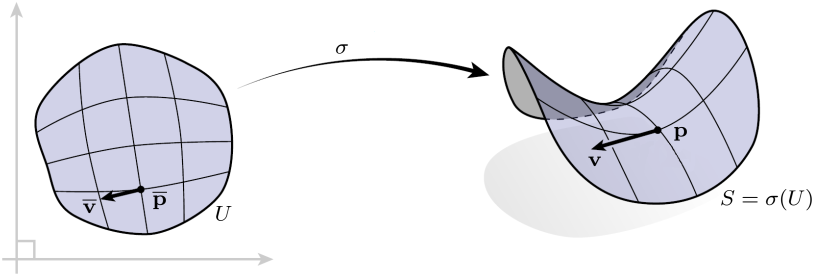
Definition (Parametrized Surface)

A parametrized surface is a mapping of class at least C^1

$$\sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3,$$

where $U \subset \mathbb{R}^2$ is a domain. We will call the image $S = \sigma(U) \subset \mathbb{R}^3$ a surface.

- The mapping σ is of class C^k if it has continuous partial derivatives up to order k.
- The mapping σ assigns to points $\overline{\mathbf{p}} = [u, v] \in U$ points $\mathbf{p} = \sigma(\overline{\mathbf{p}}) = [x, y, z] \in S$.
- The pair $[u, v] \in U$ is called the local coordinates of the point $\mathbf{p} \in S$ w.r.t. σ.



Curve on a Surface

- Let a parametrized surface be given

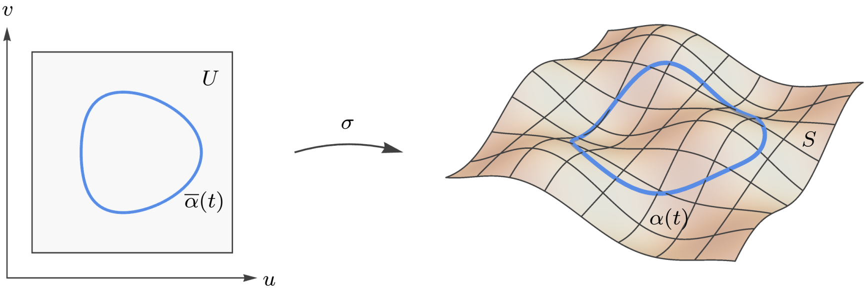
$$\sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3.$$

- Consider a curve in the parameter domain

$$\overline{\alpha} : I \to U, \qquad \overline{\alpha}(t) = [u(t), v(t)].$$

- Its image on the surface is obtained by composing the mappings:

$$\alpha(t) = (\sigma \circ \overline{\alpha})(t) = \sigma(u(t), v(t)).$$



Regular Parametrized Surface

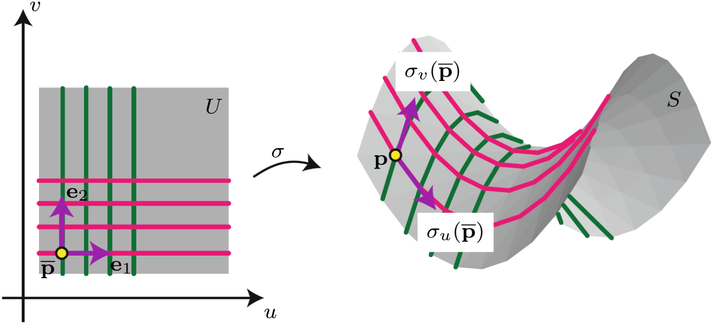
Definition (Regular Parametrized Surface)

A surface $S = \sigma(U)$ is called regular if at every point $\overline{\mathbf{p}} \in U$ the vectors

$$\sigma_u(\overline{\mathbf{p}}) \quad \text{and} \quad \sigma_v(\overline{\mathbf{p}})$$

are linearly independent, where σ_u and σ_v are the partial derivatives of σ with respect to u and v, respectively.

- The vectors σ_u, σ_v give tangent directions at the point.
- The curves on S corresponding to $u =$ const. and $v =$ const. are called parameter curves.

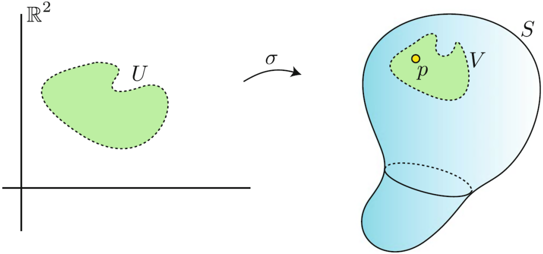


A More General Definition of a Surface (We Will Not Use)

- In general, a regular surface $S \subset \mathbb{R}^3$ is defined so that every point $\mathbf{p} \in S$ has a neighborhood $V \subset S$ that is mapped smoothly bijectively (diffeomorphically) onto some open set $U \subset \mathbb{R}^2$:

  $$\sigma : U \to V.$$

  We call such a mapping a chart.



- A single parametrization is usually not enough – for example, the sphere cannot be covered by one smooth chart without degeneracy (e.g., at the poles).
- General surfaces therefore require several charts that overlap smoothly. The collection of all such charts forms an atlas of the surface.
- In this course, however, we will always work with a single global parametrization:

  $$\sigma : U \to S, \qquad S = \sigma(U).$$

Tangent Space

Definition (Tangent Space via Curves)

Let $S \subset \mathbb{R}^3$ be a regular surface. Then the tangent space of S at a point $\mathbf{p} \in S$ is

$$T_{\mathbf{p}}S = \left\{ \gamma'(0) \;\middle|\; \gamma \text{ is a regular curve on } S,\ \gamma(0) = \mathbf{p} \right\}.$$

Lemma

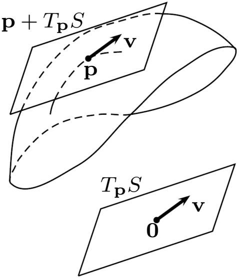
If $S = \sigma(U)$ is a regular surface and $\mathbf{p} = \sigma(\overline{\mathbf{p}})$, then

$$T_{\mathbf{p}}S = \operatorname{span}\{\sigma_u(\overline{\mathbf{p}}),\, \sigma_v(\overline{\mathbf{p}})\}.$$

- It is a two-dimensional vector subspace of the direction space $\mathbb{R}^3$ of the affine space $\mathbb{R}^3$.
- Every tangent vector $\mathbf{v} \in T_{\mathbf{p}}S$ can be expressed as

$$\mathbf{v} = v_1\, \sigma_u(\overline{\mathbf{p}}) + v_2\, \sigma_v(\overline{\mathbf{p}}), \qquad v_1, v_2 \in \mathbb{R}.$$

- The tangent plane is then the affine plane $\mathbf{p} + T_{\mathbf{p}}S$, obtained by translating the tangent space $T_{\mathbf{p}}S$ to the point $\mathbf{p}$.



Tangent Plane

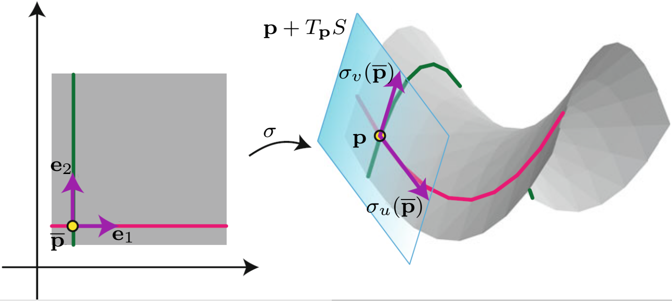
Definition (Tangent Plane)

The tangent plane of a surface S at a point $\mathbf{p}$ is the affine plane obtained by translating the tangent space to the point $\mathbf{p}$:

$$\mathbf{p} + T_{\mathbf{p}}S = \{\, \mathbf{p} + \mathbf{v} \; ; \; \mathbf{v} \in T_{\mathbf{p}}S \,\}.$$

- It is an affine subspace of the affine space $\mathbb{R}^3$.
- It can be written parametrically as

$$\mathbf{x}(u,v) = \mathbf{p} + u\,\sigma_u(\overline{\mathbf{p}}) + v\,\sigma_v(\overline{\mathbf{p}}), \qquad u, v \in \mathbb{R}.$$



Differential

- The set $U \subset \mathbb{R}^2$ can be viewed as a regular surface, and for every point $\overline{\mathbf{p}} \in U$ we have: $T_{\overline{\mathbf{p}}}U \cong \mathbb{R}^2$.

Definition (Differential of a Parametrization)

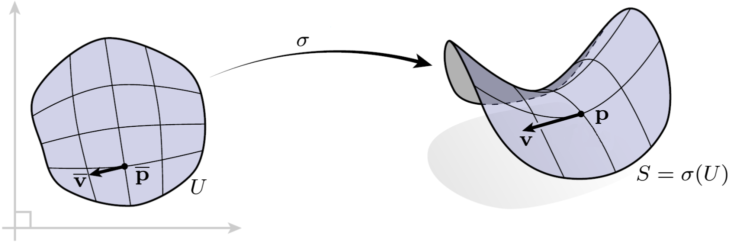
Consider a parametrized surface

$$\sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3, \qquad \overline{\mathbf{p}} \mapsto \mathbf{p} = \sigma(\overline{\mathbf{p}}).$$

The differential at the point $\mathbf{p} = \sigma(\overline{\mathbf{p}})$ is the linear mapping

$$\mathrm{d}\sigma_{\mathbf{p}} : T_{\overline{\mathbf{p}}}U \to T_{\mathbf{p}}S, \qquad \overline{\mathbf{v}} \mapsto \mathbf{v} = \mathrm{d}\sigma_{\mathbf{p}}(\overline{\mathbf{v}}).$$

- The differential maps “vectors in the parameter plane” to tangent vectors on the surface.



Differential – Jacobian Matrix

Definition (Matrix of the Differential)

The differential $\mathrm{d}\sigma_{\mathbf{p}}$ of the parametrization

$$\sigma(u,v) = [x(u,v), y(u,v), z(u,v)]$$

at the point $\mathbf{p} = \sigma(\overline{\mathbf{p}})$ is represented in the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ by the Jacobian matrix

$$\mathbf{J}_\sigma(\overline{\mathbf{p}}) = \begin{pmatrix} \frac{\partial x}{\partial u}(\overline{\mathbf{p}}) & \frac{\partial x}{\partial v}(\overline{\mathbf{p}}) \\ \frac{\partial y}{\partial u}(\overline{\mathbf{p}}) & \frac{\partial y}{\partial v}(\overline{\mathbf{p}}) \\ \frac{\partial z}{\partial u}(\overline{\mathbf{p}}) & \frac{\partial z}{\partial v}(\overline{\mathbf{p}}) \end{pmatrix}.$$

- Recall that the columns of this matrix $\sigma_u(\overline{\mathbf{p}})$ and $\sigma_v(\overline{\mathbf{p}})$ span the tangent space $T_{\mathbf{p}}S$.
- A vector $\overline{\mathbf{v}} \in T_{\overline{\mathbf{p}}}U$ is mapped to $\mathbf{v} \in T_{\mathbf{p}}S$ by

  $$\mathbf{v} = \mathbf{J}_\sigma(\overline{\mathbf{p}})\,\overline{\mathbf{v}},$$

  where vectors are always treated as column vectors.
- The coordinates of $\overline{\mathbf{v}}$ are the coordinates of the vector $\mathbf{v}$ in the basis $\{\sigma_u(\overline{\mathbf{p}}), \sigma_v(\overline{\mathbf{p}})\}$.

Proposition

A parametrized surface $\sigma : U \to \mathbb{R}^3$ is regular at the point $\mathbf{p} = \sigma(\overline{\mathbf{p}})$ if the Jacobian matrix $\mathbf{J}_\sigma(\overline{\mathbf{p}})$ has rank 2.

Example: Differential

Example Tangent Plane and Differential)

Consider the parametrization of a paraboloid

$$\sigma(u,v) = [u, v, u^2 + v^2], \qquad u, v \in \mathbb{R}.$$

- The point with local coordinates $\overline{\mathbf{p}} = [1, 2]$ maps to
$$\mathbf{p} = \sigma(\overline{\mathbf{p}}) = [1, 2, 5].$$
- Partial derivatives at $\overline{\mathbf{p}}$ generate the tangent plane
$$\sigma_u(\overline{\mathbf{p}}) = (1, 0, 2), \qquad \sigma_v(\overline{\mathbf{p}}) = (0, 1, 4).$$
- The differential $\mathrm{d}\sigma_{\mathbf{p}}$ is described by the Jacobian matrix
$$\mathbf{J}_\sigma(\overline{\mathbf{p}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 4 \end{pmatrix}.$$
- The vector $\overline{\mathbf{v}} = (1, -1) \in T_{\overline{\mathbf{p}}}U$ maps to
$$\mathbf{v} = \mathrm{d}\sigma_{\mathbf{p}}(\overline{\mathbf{v}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

Normal Vector to a Surface

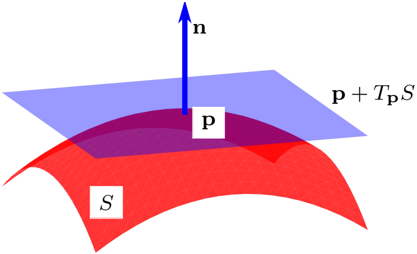
Definition (Normal Vector)

A vector $\mathbf{n} \in \mathbb{R}^3$ is called a normal vector to the surface S at a point $\mathbf{p} \in S$ if

$$\mathbf{n} \cdot \mathbf{v} = 0 \quad \text{for all } \mathbf{v} \in T_{\mathbf{p}}S.$$

- A unit normal vector is a normal vector of length 1.
- If $S = \sigma(U)$ is regular and $\mathbf{p} = \sigma(\overline{\mathbf{p}})$, then a unit normal vector at $\mathbf{p}$ is

$$\mathbf{n} = \frac{\sigma_u(\overline{\mathbf{p}}) \times \sigma_v(\overline{\mathbf{p}})}{\|\sigma_u(\overline{\mathbf{p}}) \times \sigma_v(\overline{\mathbf{p}})\|}.$$



Vector Fields on a Surface

Definition (Vector Field on a Surface)

A vector field on a regular surface S is a mapping of class at least C^1

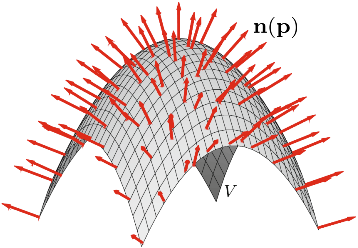
$$\mathbf{v} : S \to \mathbb{R}^3.$$

It is called:

- a tangent field if $\mathbf{v}(\mathbf{p}) \in T_{\mathbf{p}}S$,
- a normal field if $\mathbf{v}(\mathbf{p})$ is a normal vector to S at $\mathbf{p}$,

for all $\mathbf{p} \in S$.

- A unit normal field is typically denoted by $\mathbf{n}(\mathbf{p})$.



Orientation of a Surface

Definition (Orientation of a Surface)

An orientation of a regular surface means a unit normal vector field defined on it.

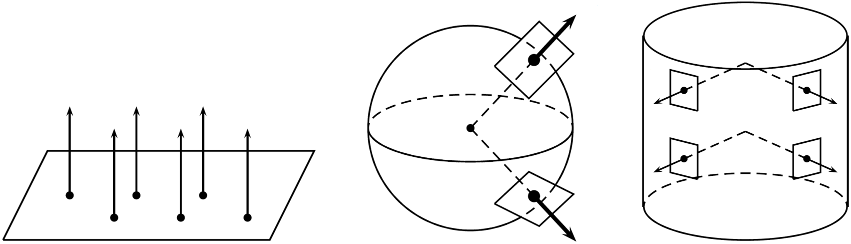
- A surface S is called orientable if an orientation exists.
- An orientable surface together with a chosen orientation is called an oriented surface.

- The orientation determines which side of the surface is considered “upper”.
- Each unit normal vector can be viewed as an arrow pointing “outward”.
- Every orientable surface has exactly two orientations, given by $\mathbf{n}$ or $-\mathbf{n}$.

Examples of Orientable Surfaces

Example (Plane, Sphere, Cylinder)

- The xy-plane has a constant normal field $\mathbf{n}(\mathbf{p}) = (0, 0, 1)$.
- The sphere S^2 has the natural normal field $\mathbf{n}(\mathbf{p}) = \mathbf{p}$.
- The cylinder $S^1 \times \mathbb{R}$ has the normal field $\mathbf{n}(x, y, z) = (x, y, 0)$.

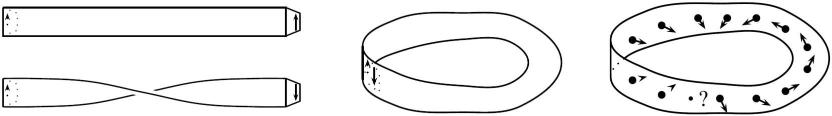


Non-Orientability of the Möbius Strip

Example

The Möbius strip S is not orientable.

- There is no continuous unit normal field defined on the entire strip.
- If we try to define a normal vector and travel once around the strip, the orientation of the vector reverses.
- The Möbius strip has only one side and one boundary, which makes a global choice of unit normal impossible – hence no global orientation.



- Orientation of a surface is necessary, for example, for defining the second fundamental form, principal curvatures, and mean curvature.

First Fundamental Form

Definition (First Fundamental Form)

The first fundamental form at a point $\mathbf{p} \in S$ is the bilinear form

$$\mathrm{I}_{\mathbf{p}} : T_{\mathbf{p}}S \times T_{\mathbf{p}}S \to \mathbb{R},$$

which assigns to each pair of vectors $\mathbf{u}, \mathbf{v} \in T_{\mathbf{p}}S$ their inner product:

$$\mathrm{I}_{\mathbf{p}}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}.$$

- For $\mathbf{u}, \mathbf{v} \in T_{\mathbf{p}}S$, this is the restriction of the inner product from $\mathbb{R}^3$ to the tangent plane $T_{\mathbf{p}}S$ – computed exactly the same way as in $\mathbb{R}^3$.
- A measurement on a surface is called intrinsic if it can be expressed solely using the first fundamental form.
- Geometrically: we measure only quantities on the surface itself (lengths, angles, areas), independently of how it is embedded in $\mathbb{R}^3$.

Proposition

The first fundamental form is symmetric:

$$\mathrm{I}_{\mathbf{p}}(\mathbf{u}, \mathbf{v}) = \mathrm{I}_{\mathbf{p}}(\mathbf{v}, \mathbf{u}).$$

First Fundamental Form in Local Coordinates

- Let two tangent vectors of the surface $S = \sigma(U)$ at a point $\mathbf{p}$ be given by their local coordinates $\overline{\mathbf{u}}, \overline{\mathbf{v}} \in \mathbb{R}^2$.
- The actual tangent vectors in $T_{\mathbf{p}}S$ are obtained using the Jacobian matrix:
$$\mathbf{u} = \mathbf{J}_\sigma\, \overline{\mathbf{u}}, \qquad \mathbf{v} = \mathbf{J}_\sigma\, \overline{\mathbf{v}}.$$
- The first fundamental form $\mathrm{I}_{\mathbf{p}}$ measures the inner product:
$$\mathrm{I}_{\mathbf{p}}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} = (\mathbf{J}_\sigma \overline{\mathbf{u}})^\top (\mathbf{J}_\sigma \overline{\mathbf{v}}) = \overline{\mathbf{u}}^\top \underbrace{\mathbf{J}_\sigma^\top \mathbf{J}_\sigma}_{\mathbf{I}}\, \overline{\mathbf{v}}.$$
- The matrix
$$\mathbf{I} = \mathbf{J}_\sigma^\top \mathbf{J}_\sigma = \begin{pmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_u \cdot \sigma_v & \sigma_v \cdot \sigma_v \end{pmatrix}$$
is the Gram matrix of the vectors σ_u, σ_v and represents the inner product in local coordinates.

Proposition (Matrix of the First Fundamental Form)

The first fundamental form $\mathrm{I}_{\mathbf{p}}$ of the surface σ at $\mathbf{p} = \sigma(\overline{\mathbf{p}})$ is given with respect to the basis $\{\sigma_u, \sigma_v\}$ by the matrix
$$\mathbf{I}(\overline{\mathbf{p}}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$
where
$$E = \sigma_u \cdot \sigma_u, \quad F = \sigma_u \cdot \sigma_v, \quad G = \sigma_v \cdot \sigma_v.$$

Computations Using the First Fundamental Form

- The first fundamental form allows us to measure lengths, angles, and areas on the surface using only data from the parameter domain.

Proposition (Computations via the First Fundamental Form)

Let $\alpha(t) = \sigma(\overline{\alpha}(t)),\ t \in I$ and $\beta(s) = \sigma(\overline{\beta}(s)),\ s \in J$ be curves on the surface $S = \sigma(U)$ such that $\alpha(t_0) = \beta(s_0) = \mathbf{p}$. Then:

- Length of the curve α:

$$L = \int_I \sqrt{\overline{\alpha}'(t)^\top \mathbf{I}(\overline{\alpha}(t))\overline{\alpha}'(t)}\,\mathrm{d}t$$

- Angle between curves α, β at the point $\mathbf{p}$:

$$\cos\varphi = \frac{\overline{\alpha}'(t_0)^\top \mathbf{I}(\overline{\mathbf{p}})\overline{\beta}'(s_0)}{\sqrt{\overline{\alpha}'(t_0)^\top \mathbf{I}(\overline{\mathbf{p}})\overline{\alpha}'(t_0)}\sqrt{\overline{\beta}'(s_0)^\top \mathbf{I}(\overline{\mathbf{p}})\overline{\beta}'(s_0)}}$$

- Area of the image of a region $D \subset U$ on the surface:

$$\mathrm{Area}(\sigma(D)) = \iint_D \sqrt{\det \mathbf{I}(u,v)}\,\mathrm{d}u\,\mathrm{d}v$$

- All these computations are intrinsic – independent of the embedding of the surface in space.

Example: Curve Length on a Paraboloid

Example (Paraboloid)

Consider the parametrization $\sigma(u,v)$ and the curve $\alpha(t) = \sigma(\overline{\alpha}(t))$, where

$$\sigma(u,v) = [u, v, u^2 + v^2], \quad u, v \in \mathbb{R}, \qquad \overline{\alpha}(t) = [t, t], \quad t \in [0,1].$$

- Partial derivatives:
$$\sigma_u = (1, 0, 2u), \quad \sigma_v = (0, 1, 2v).$$
- Matrix of the first fundamental form:
$$\mathbf{I}(u,v) = \begin{pmatrix} 1 + 4u^2 & 4uv \\ 4uv & 1 + 4v^2 \end{pmatrix}.$$
- 1) Curve length via the first fundamental form:
$$L = \int_0^1 \sqrt{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{I}(t,t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \, \mathrm{d}t = \int_0^1 \sqrt{2 + 16t^2} \, \mathrm{d}t \approx 1{,}864.$$
- 2) As a spatial (3D) curve:
$$\alpha'(t) = (1, 1, 4t), \quad \|\alpha'(t)\| = \sqrt{2 + 16t^2},$$

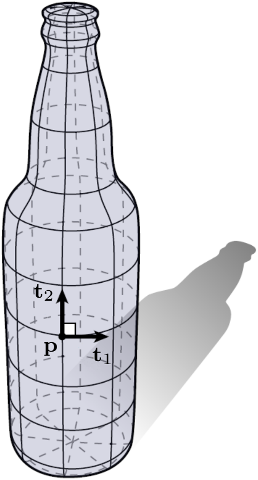
which gives the same integral.

Motivation: Normal Curvature

- When we speak about the curvature of a surface, we usually have an intuitive idea, e.g.

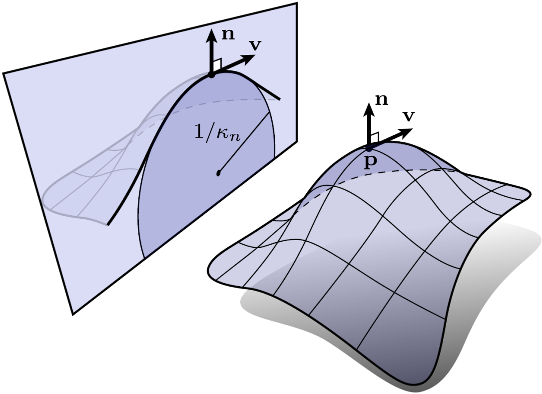
  a ball is curved, vs. a table is flat.

- But consider a beer bottle:
  - In one direction (around the bottle) it is curved,
  - in another direction (along the bottle) it is flat.
- Therefore, on a surface it is not enough to say "it has curvature"; we must measure curvature in a chosen direction at a given point.
- This concept is called normal curvature.
- From normal curvature we derive other curvatures such as Gaussian and mean curvature.



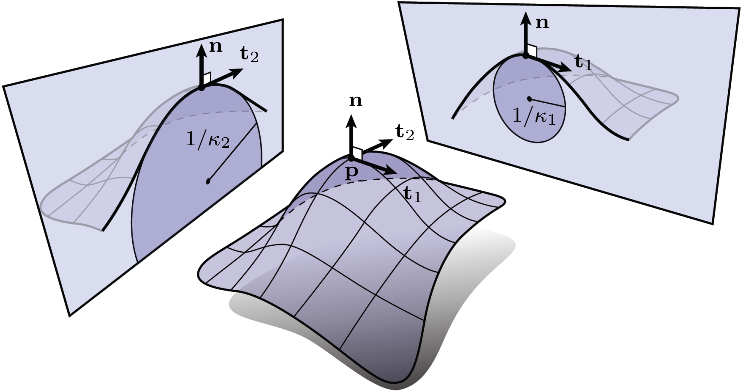
What is Normal Curvature

- Consider the normal plane of the surface S at a point $\mathbf{p}$, determined by
  - the point $\mathbf{p}$ on the surface,
  - a vector $\mathbf{v} \in T_{\mathbf{p}}S$ (direction in the tangent plane),
  - the normal vector $\mathbf{n}$ of the surface at $\mathbf{p}$.
- The intersection of this plane with the surface is a planar curve whose signed curvature at $\mathbf{p}$ is called the normal curvature $\kappa_n(\mathbf{v})$.



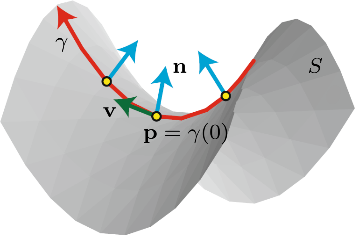
Principal Directions and Principal Curvatures

- At each point $\mathbf{p}$ we may ask: in which directions does the surface bend the most and the least?
- The directions where the normal curvature is maximal and minimal are called the principal directions $\mathbf{t}_1$, $\mathbf{t}_2$.
- The corresponding curvatures are called the principal curvatures κ_1, κ_2.



How to Compute Normal Curvature

- Recall that for a unit-speed curve γ in $\mathbb{R}^n$ we have
$$\kappa = -\mathbf{n}' \cdot \mathbf{t},$$
where $\mathbf{t}$ is the unit tangent vector and $\mathbf{n}$ is the unit normal vector along the curve.
- To generalize this idea to surfaces, we replace the curve normal vector by the unit normal field of the surface.
- For $\mathbf{p} \in S$ and $\mathbf{v} \in T_{\mathbf{p}}S$, consider a curve $\gamma(t) \subset S$ with $\gamma(0) = \mathbf{p}$, $\gamma'(0) = \mathbf{v}$.
- We track the change of the surface normal $\mathbf{n}(\gamma(t))$ along the curve.
- We will show that the normal curvature in direction $\mathbf{v}$ is
$$\kappa_n(\mathbf{v}) = -\mathbf{n}'(\gamma(0)) \cdot \mathbf{v}.$$
- In practice, we use the Weingarten map, which corresponds to the derivative of the Gauss map.



Gauss Map

- Let S be an oriented surface with unit normal field:

$$\mathbf{n} : S \to \mathbb{R}^3.$$

- Since $\mathbf{n}(\mathbf{p})$ has unit length, its image lies on the unit sphere:

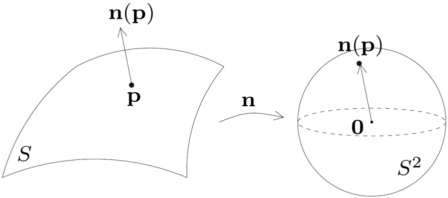
$$\mathbf{n}(S) \subset S^2 \subset \mathbb{R}^3.$$

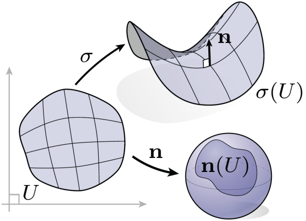
Definition (Gauss Map)

The Gauss map of a surface S is its unit normal field:

$$\mathbf{n} : S \to S^2 \subset \mathbb{R}^3.$$

- We may visualize it by translating all vectors $\mathbf{n}(\mathbf{p})$ to the origin and viewing them as points on S^2.





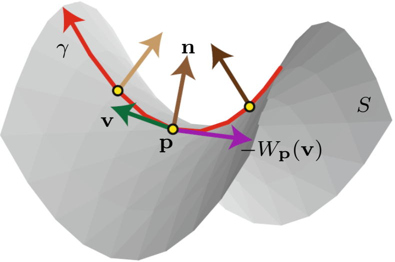
Weingarten Map

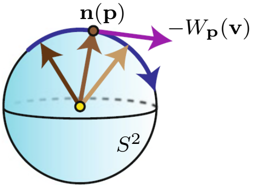
- Using the identification $T_{\mathbf{p}}S \cong T_{\mathbf{n}(\mathbf{p})}S^2$, the derivative of the Gauss map can be viewed as
$$\mathrm{d}\mathbf{n}_{\mathbf{p}} : T_{\mathbf{p}}S \to T_{\mathbf{p}}S.$$

Definition (Weingarten Map)

The Weingarten map of the surface S at a point $\mathbf{p}$ is the linear mapping
$$W_{\mathbf{p}} = -\mathrm{d}\mathbf{n}_{\mathbf{p}} : T_{\mathbf{p}}S \to T_{\mathbf{p}}S.$$

- The minus sign in $-\mathrm{d}\mathbf{n}_{\mathbf{p}}$ appears because the change of the normal vector has the opposite direction to the bending of the surface.





Second Fundamental Form

- The Weingarten map $W_{\mathbf{p}} : T_{\mathbf{p}}S \to T_{\mathbf{p}}S$ describes how the normal vector changes along the surface.
- The second fundamental form $\mathrm{II}_{\mathbf{p}}$ captures the variation of the normal using the Weingarten map.

Definition (Second Fundamental Form)

The second fundamental form of the surface S at a point $\mathbf{p} \in S$ is the bilinear form

$$\mathrm{II}_{\mathbf{p}} : T_{\mathbf{p}}S \times T_{\mathbf{p}}S \to \mathbb{R},$$

which assigns to each pair of vectors $\mathbf{u}, \mathbf{v} \in T_{\mathbf{p}}S$ the value

$$\mathrm{II}_{\mathbf{p}}(\mathbf{u}, \mathbf{v}) = W_{\mathbf{p}}(\mathbf{u}) \cdot \mathbf{v} = -\mathrm{d}\mathbf{n}_{\mathbf{p}}(\mathbf{u}) \cdot \mathbf{v}.$$

Second Fundamental Form in Local Coordinates

- For a parametrization $\sigma(u,v)$ and normal field $\mathbf{n}(u,v)$ we have

$$\partial_u(\sigma_u \cdot \mathbf{n}) = 0 \longrightarrow (\partial_u \sigma_u) \cdot \mathbf{n} + \sigma_u \cdot (\partial_u \mathbf{n}) = 0 \longrightarrow \sigma_{uu} \cdot \mathbf{n} = -\mathbf{n}_u \cdot \sigma_u .$$

- Similarly we obtain

$$\begin{aligned} e &= W_{\mathbf{p}}(\sigma_u) \cdot \sigma_u = -\mathbf{n}_u \cdot \sigma_u = \sigma_{uu} \cdot \mathbf{n}, \\ f &= W_{\mathbf{p}}(\sigma_u) \cdot \sigma_v = -\mathbf{n}_u \cdot \sigma_v = -\mathbf{n}_v \cdot \sigma_u = \sigma_{uv} \cdot \mathbf{n}, \\ g &= W_{\mathbf{p}}(\sigma_v) \cdot \sigma_v = -\mathbf{n}_v \cdot \sigma_v = \sigma_{vv} \cdot \mathbf{n}. \end{aligned}$$

Proposition (Matrix of the Second Fundamental Form)

The second fundamental form $\mathrm{II}_{\mathbf{p}}$ of the surface σ at $\mathbf{p} = \sigma(\overline{\mathbf{p}})$ is given with respect to the basis $\{\sigma_u, \sigma_v\}$ by

$$\mathbf{II}(\overline{\mathbf{p}}) = \begin{pmatrix} e & f \\ f & g \end{pmatrix},$$

where

$$e = \sigma_{uu} \cdot \mathbf{n}, \quad f = \sigma_{uv} \cdot \mathbf{n}, \quad g = \sigma_{vv} \cdot \mathbf{n}.$$

Corollary (Symmetry of the Second Fundamental Form)

The second fundamental form is symmetric:

$$\mathrm{II}_{\mathbf{p}}(\mathbf{u}, \mathbf{v}) = \mathrm{II}_{\mathbf{p}}(\mathbf{v}, \mathbf{u}).$$

Weingarten Map in Local Coordinates

- We have

$$\overline{\mathbf{u}}^\top \mathbf{II}\,\overline{\mathbf{v}} = \underbrace{\mathrm{II}_{\mathbf{p}}(\mathbf{u},\mathbf{v}) = \mathrm{II}_{\mathbf{p}}(\mathbf{v},\mathbf{u})}_{\text{symmetry of } \mathrm{II}_{\mathbf{p}}} = \underbrace{W_{\mathbf{p}}(\mathbf{v})\cdot\mathbf{u} = \mathbf{u}\cdot W_{\mathbf{p}}(\mathbf{v})}_{\text{symmetry of } \mathrm{I}_{\mathbf{p}}} = \overline{\mathbf{u}}^\top \mathbf{I}\,\mathbf{W}\,\overline{\mathbf{v}}.$$

- Hence

$$\mathbf{II} = \mathbf{I}\,\mathbf{W}, \qquad \mathbf{W} = \mathbf{I}^{-1}\,\mathbf{II}.$$

- The matrix $\mathbf{W}$ represents the Weingarten map with respect to the basis $\{\sigma_u, \sigma_v\}$.

Proposition (Matrix of the Weingarten Map)

The Weingarten map $W_{\mathbf{p}}$ of the surface σ at $\mathbf{p} = \sigma(\overline{\mathbf{p}})$ is represented by

$$\mathbf{W}(\overline{\mathbf{p}}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} Ge - Ff & Gf - Fg \\ -Fe + Ef & -Ff + Eg \end{pmatrix},$$

where

$$E = \sigma_u \cdot \sigma_u, \quad F = \sigma_u \cdot \sigma_v, \quad G = \sigma_v \cdot \sigma_v, \quad e = \sigma_{uu} \cdot \mathbf{n}, \quad f = \sigma_{uv} \cdot \mathbf{n}, \quad g = \sigma_{vv} \cdot \mathbf{n}.$$

Normal Curvature

Definition (Normal Curvature)

The normal curvature in direction $\mathbf{v} \in T_{\mathbf{p}}S$ is defined as

$$\kappa_n(\mathbf{v}) = \frac{\mathrm{II}_{\mathbf{p}}(\mathbf{v},\mathbf{v})}{\mathrm{I}_{\mathbf{p}}(\mathbf{v},\mathbf{v})}.$$

- In the basis $\{\sigma_u, \sigma_v\}$:

$$\kappa_n(\overline{\mathbf{v}}) = \frac{\overline{\mathbf{v}}^\top \mathbf{II}\,\overline{\mathbf{v}}}{\overline{\mathbf{v}}^\top \mathbf{I}\,\overline{\mathbf{v}}} = \frac{\overline{\mathbf{v}}^\top \mathbf{IW}\,\overline{\mathbf{v}}}{\overline{\mathbf{v}}^\top \mathbf{I}\,\overline{\mathbf{v}}}.$$

- Where $\mathbf{v} = \mathrm{d}\sigma_{\mathbf{p}}(\overline{\mathbf{v}})$ and

$$\kappa_n(\mathbf{v}) = \frac{-\mathrm{d}\mathbf{n}_{\mathbf{p}}(\overline{\mathbf{v}}) \cdot \mathbf{v}}{\|\mathbf{v}\|^2}.$$

- If $\|\mathbf{v}\| = 1$, then

$$\kappa_n(\mathbf{v}) = -\mathrm{d}\mathbf{n}_{\mathbf{p}}(\overline{\mathbf{v}}) \cdot \mathbf{v}.$$

Meusnier's Theorem

Theorem (Meusnier's Theorem)

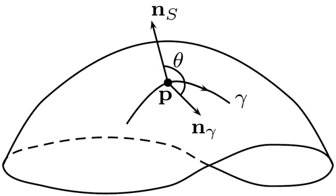
For a curve γ on an oriented surface S at the point $\gamma(0)$ we have

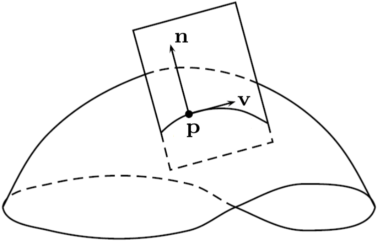
$$\kappa_n = \kappa \cos \theta,$$

where κ_n is the normal curvature of S in the direction $\gamma'(0)$, κ is the curvature of γ, and θ is the angle between the curve normal and the surface normal.

- For a curve lying in the normal plane of the surface we obtain

$$\kappa_n = \kappa.$$





Principal Curvatures and Principal Directions

Definition (Principal Curvatures and Directions)

Let S be a regular surface, $\mathbf{p}$ a point, and

$$W_{\mathbf{p}} : T_{\mathbf{p}}S \to T_{\mathbf{p}}S$$

the Weingarten map.

- The principal curvatures κ_1, κ_2 are the eigenvalues of $W_{\mathbf{p}}$.
- The principal directions $\mathbf{t}_1, \mathbf{t}_2$ are eigenvectors satisfying

$$W_{\mathbf{p}}(\mathbf{t}_i) = \kappa_i \mathbf{t}_i.$$

- If the Weingarten map is represented by the matrix $\mathbf{W}$, then
  - its eigenvalues are the principal curvatures κ_1, κ_2
  - its eigenvectors $\overline{\mathbf{t}}_1, \overline{\mathbf{t}}_2$ give the coordinates of the principal directions in the basis $\{\sigma_u(\overline{\mathbf{p}}), \sigma_v(\overline{\mathbf{p}})\}$, i.e.

$$\mathbf{t}_i = \mathbf{J}_\sigma(\overline{\mathbf{p}})\,\overline{\mathbf{t}}_i, \quad i = 1, 2.$$

Orthogonality of Principal Directions

Proposition (Orthogonality of Principal Directions)

The principal directions $\mathbf{t}_1, \mathbf{t}_2$ are orthogonal in $T_{\mathbf{p}}S$.

Proof:

- The second fundamental form is symmetric:

$$\mathrm{II}(\mathbf{u}, \mathbf{v}) = W(\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot W(\mathbf{v}).$$

- For eigenvectors we have:

$$W(\mathbf{t}_1) \cdot \mathbf{t}_2 = \kappa_1 \,(\mathbf{t}_1 \cdot \mathbf{t}_2), \quad \mathbf{t}_1 \cdot W(\mathbf{t}_2) = \kappa_2 \,(\mathbf{t}_1 \cdot \mathbf{t}_2).$$

- Substituting into the symmetry relation gives:

$$\kappa_1 \,(\mathbf{t}_1 \cdot \mathbf{t}_2) = \kappa_2 \,(\mathbf{t}_1 \cdot \mathbf{t}_2).$$

- Subtracting:

$$(\kappa_1 - \kappa_2)\,(\mathbf{t}_1 \cdot \mathbf{t}_2) = 0.$$

- If $\kappa_1 \neq \kappa_2$, then

$$\mathbf{t}_1 \cdot \mathbf{t}_2 = 0.$$

□

Euler's Identity

Proposition (Euler's Identity)

For any unit vector $\mathbf{v}_\varphi = \cos\varphi\,\mathbf{t}_1 + \sin\varphi\,\mathbf{t}_2 \in T_\mathbf{p}S$ we have

$$\kappa_n(\mathbf{v}_\varphi) = \kappa_1 \cos^2\varphi + \kappa_2 \sin^2\varphi.$$

Proof:

- Let $\mathbf{t}_1, \mathbf{t}_2$ be orthonormal principal directions:

$$W_\mathbf{p}(\mathbf{t}_1) = \kappa_1 \mathbf{t}_1, \quad W_\mathbf{p}(\mathbf{t}_2) = \kappa_2 \mathbf{t}_2.$$

- Any unit vector in direction φ can be written as

$$\mathbf{v}_\varphi = \cos\varphi\,\mathbf{t}_1 + \sin\varphi\,\mathbf{t}_2.$$

- Normal curvature equals

$$\kappa_n(\mathbf{v}_\varphi) = \mathrm{II}(\mathbf{v}_\varphi, \mathbf{v}_\varphi) = W_\mathbf{p}(\mathbf{v}_\varphi) \cdot \mathbf{v}_\varphi.$$

- After expanding and using orthogonality $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$ we obtain

$$\kappa_n(\mathbf{v}_\varphi) = \kappa_1 \cos^2\varphi + \kappa_2 \sin^2\varphi.$$

☐

Principal Curvatures as Extremal Normal Curvatures

Proposition

The principal curvatures κ_1, κ_2 are the maximum and minimum values of the normal curvature.

Proof:

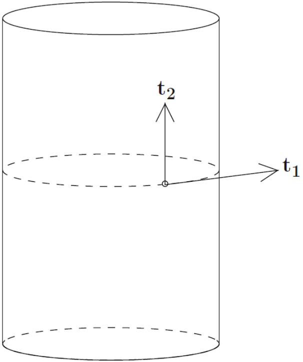
- Euler's identity:
  $$\kappa_n(\varphi) = \kappa_1 \cos^2\varphi + \kappa_2 \sin^2\varphi,$$
  where φ is the angle with respect to the principal direction of κ_1.
- Derivative with respect to φ:
  $$\frac{d}{d\varphi}\kappa_n(\varphi) = 2(\kappa_2 - \kappa_1)\sin\varphi\cos\varphi.$$
- Stationary points:
  $$\sin\varphi\cos\varphi = 0 \implies \varphi = 0,\ \frac{\pi}{2}.$$
- Values in these directions:
  $$\kappa_n(0) = \kappa_1, \quad \kappa_n\left(\frac{\pi}{2}\right) = \kappa_2.$$
- Hence κ_1, κ_2 are extremal normal curvatures. $\square$

Principal Directions on a Cylinder

Example (Principal Directions on a Cylinder)

On a surface of revolution cylinder the principal directions are

- direction $\mathbf{t}_1$ along the circles, where $\kappa_1 = \dfrac{1}{r}$,
- direction $\mathbf{t}_2$ along the axis, where $\kappa_2 = 0$.



Gaussian and Mean Curvature

Definition (Gaussian and Mean Curvature)

Let κ_1, κ_2 be the principal curvatures at a point $\mathbf{p} \in S$.

- Gaussian curvature:

$$K = \kappa_1 \cdot \kappa_2$$

- Mean curvature:

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

- Gaussian curvature equals the determinant of the Weingarten matrix:

$$K = \det \mathbf{W}.$$

- Mean curvature equals half of its trace:

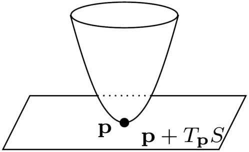
$$H = \frac{1}{2} \operatorname{tr} \mathbf{W}.$$

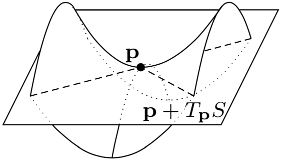
Types of Points According to Gaussian Curvature

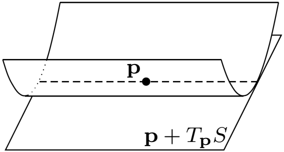
Definition (Classification of Points)

Points $\mathbf{p} \in S$ are classified according to the value of Gaussian curvature $K(\mathbf{p})$:

- elliptic point – $K > 0$,
- hyperbolic point – $K < 0$,
- parabolic point – $K = 0$.



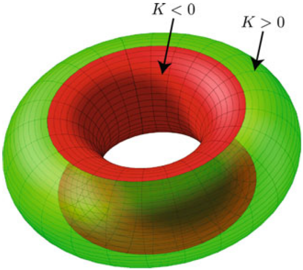


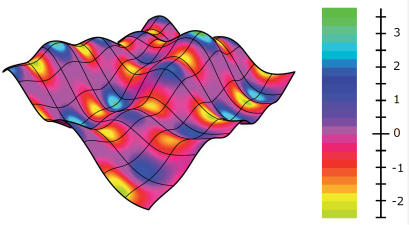


Examples of Surfaces by Point Type

Example

- A surface consisting only of elliptic points: sphere.
- A surface consisting only of hyperbolic points: hyperbolic paraboloid.
- A surface consisting only of parabolic points: cone (except the apex), cylinder.
- A surface containing all types of points: torus.





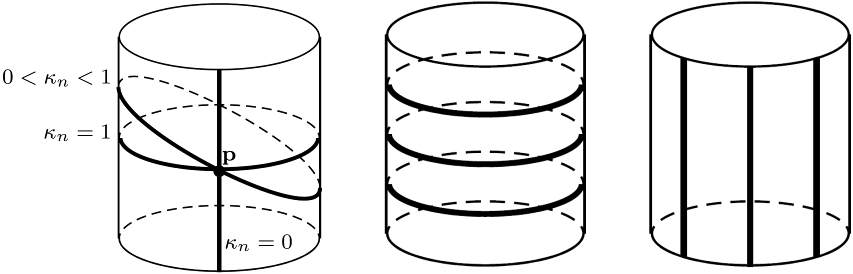
Principal Curvature Lines

Definition (Principal Curve)

A parametrized curve $\gamma : I \to S$ on an oriented regular surface S is called a principal curve if at each point the tangent vector $\gamma'(t)$ is a principal direction.

Example

Principal curves on a cylinder are straight lines (along the axis) and circles (perpendicular to the axis).



Asymptotic Directions and Curves

Definition (Asymptotic Direction)

A vector $\mathbf{v} \in T_{\mathbf{p}}S$ on an oriented regular surface S is called an asymptotic direction at $\mathbf{p}$ if

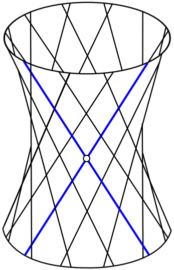
$$\kappa_n(\mathbf{v}) = 0.$$

Definition (Asymptotic Curve)

A parametrized curve $\gamma : I \to S$ on an oriented regular surface S is called an asymptotic curve if at each point the tangent vector $\gamma'(t)$ is an asymptotic direction.

Example

On a one-sheeted rotational hyperboloid, the asymptotic curves form two conjugate families of straight lines.



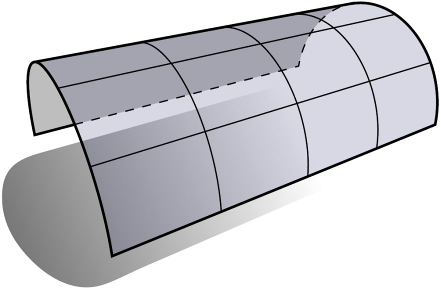
Developable Surfaces

Definition (Developable Surface)

An oriented regular surface S is called developable if its Gaussian curvature vanishes identically, i.e. $K(\mathbf{p}) = 0$ for all $\mathbf{p} \in S$.

Example

Examples of developable surfaces: cylinder, cone, tangent surface of a space curve.



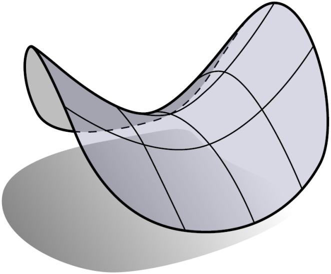
Minimal Surfaces

Definition (Minimal Surface)

An oriented regular surface S is called minimal if its mean curvature vanishes identically, i.e. $H(\mathbf{p}) = 0$ for all $\mathbf{p} \in S$.

Example

Examples of minimal surfaces: plane, helicoid.



Theorema Egregium

Theorem (Gauss' Theorema Egregium)

The Gaussian curvature of an oriented regular surface S is an intrinsic quantity: it can be expressed solely using the first fundamental form (and its derivatives), hence only through intrinsic measurements on the surface.

- Gaussian curvature is invariant under isometries – it does not depend on how the surface is bent in space, provided lengths and angles on the surface remain unchanged.
- It can be determined purely from measurements on the surface (lengths, angles), without knowing the embedding in space.

Example

For example, a part of a sphere cannot be developed onto the plane without distortion, because the sphere has positive Gaussian curvature while the plane has zero curvature.