METODY ŘEŠENÍ DIF. ROVNIC 1. ŘÁDU

- neexistuje univerzální metoda pro nalezení řešení DR
- <u>náš cíl:</u>
  - obecné řešení DR
  - řešení počáteční úlohy

- METODA PŘÍMÉ INTEGRACE

$$x'(t) = f(t) \; ; \; t \in I \qquad (1)$$

→ obecné řešení: stanovení primitivní fce k f

$$\boxed{x(t) = F(t) + C} \qquad F(t) = \int f(t)\,dt$$

→ počáteční úloha: stanovení konstanty C

$$x(t_0) = x_0 \qquad (2)$$

$$x(t_0) = F(t_0) + C \Rightarrow C = x(t_0) - F(t_0)$$

-
$$\int_{t_0}^{t} x'(t)\,dt = \int_{t_0}^{t} f(t)\,dt$$

$$x(t) - x(t_0) = \int_{t_0}^{t} f(t)\,dt$$

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tilde{t})\,d\tilde{t}$$

$\boxed{\text{PŘ}}$ $x'(t) = t^3 + \sin t$

$$x(t) = \int t^3 + \sin t \, dt = \boxed{\frac{t^4}{4} + \cos t + C}$$

$\boxed{\text{PŘ}}$

$$\begin{cases} x'(t) = t^2 + \dfrac{1}{1+2t^2} \\ x(0) = 0 \end{cases}$$

$$\int_{t_0}^{t} x'(t)\,dt = \int_{t_0}^{t} t^2 + \frac{1}{1+2t^2}\,dt$$

$$x(t) - \underset{0}{\underset{\parallel}{x(t_0)}} = \left[\frac{t^3}{3}\right]_{t_0}^{t} + \int_{t_0}^{t} \frac{1}{1+2t^2}\,dt$$

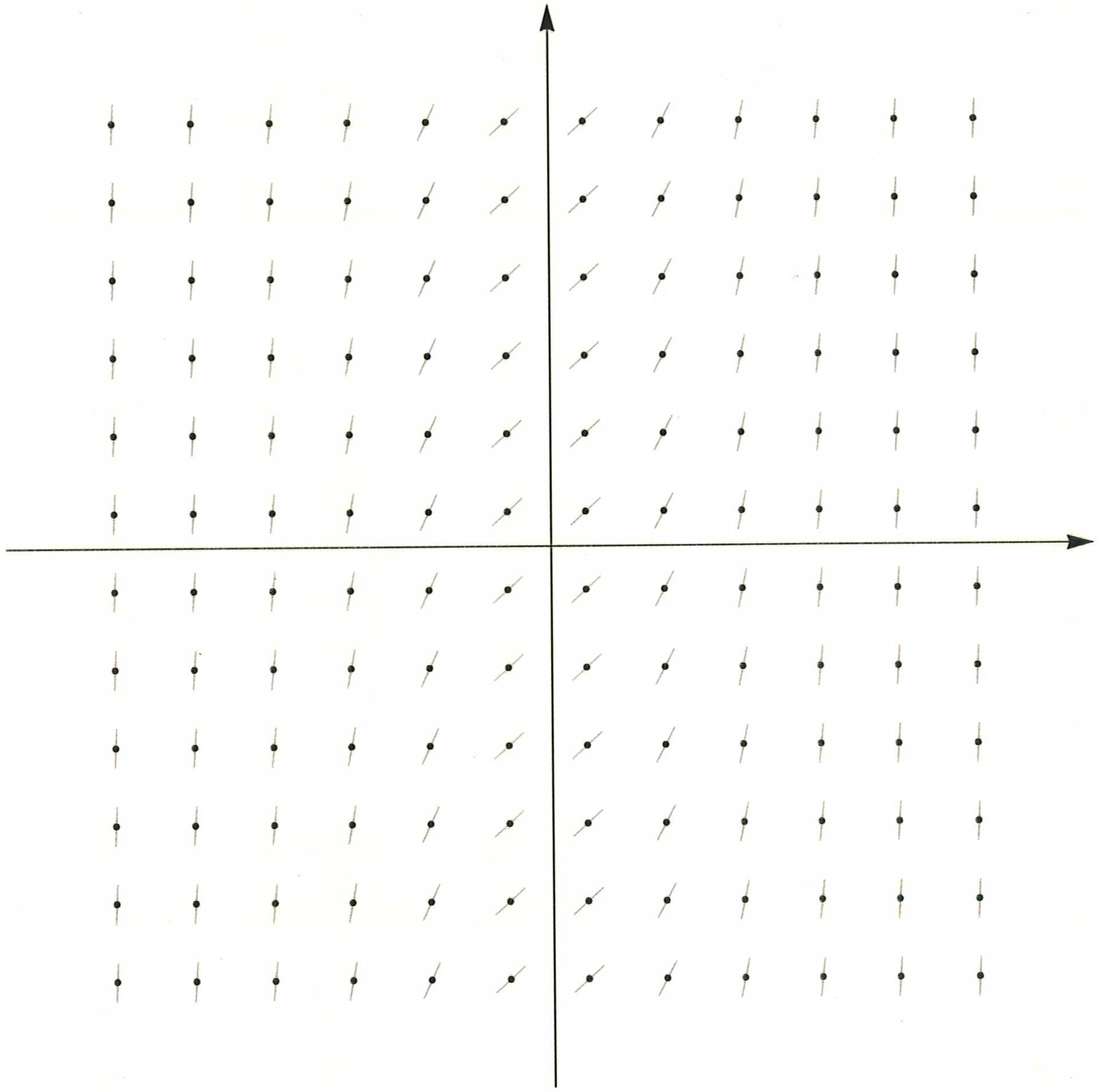
$$\int_{t_0}^{t} \frac{1}{1+2t^2}\,dt = \int_{t_0}^{t} \frac{1}{1+(\sqrt{2}t)^2}\,dt = \left\{ \begin{array}{l} \sqrt{2}t = y \\ dt = \dfrac{dy}{\sqrt{2}} \end{array} \right\} =$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{1+y^2}\,dy = \left[\frac{1}{\sqrt{2}} \operatorname{arctg}(y) \cancel{+C}\right]_{t_0}^{t}$$

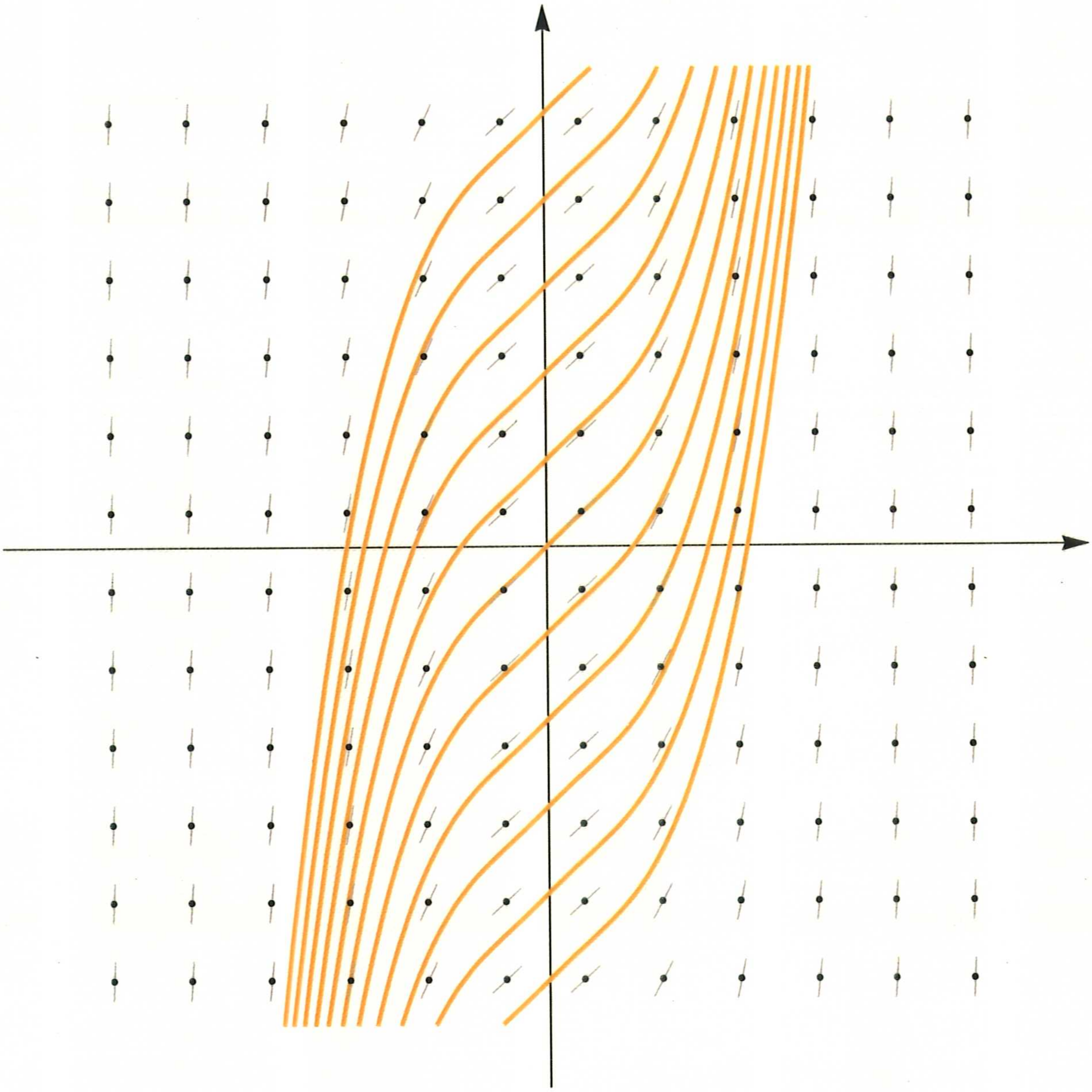
$$= \left[\frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}t)\right]_{t_0}^{t}$$

$$x(t) = x(t_0) + \left[\frac{t^3}{3} + \frac{1}{\sqrt{2}} \operatorname{arctg} \sqrt{2}t\right]_{t_0}^{t} =$$

$$= 0 + \frac{t^3}{3} + \frac{1}{\sqrt{2}} \operatorname{arctg} \sqrt{2}t$$



Obrázek 1: Směrové pole rovnice $x'(t) = t^2 + \frac{1}{1+2t^2}$



Obrázek 2: Integrální křivky rovnice $x'(t) = t^2 + \frac{1}{1+2t^2}$

ROVNICE SE SEPAROVANÝMI PROMĚNNÝMI

Nechť $f \in C(a,b)$; $g \in C(c,d)$; $g(x) \neq 0 \ \forall x \in (c,d)$,

potom řešení p.ú.

$$\begin{cases} x'(t) = f(t)\, g(x(t)) \\ x(t_0) = x_0 \end{cases} \qquad (S) \qquad p(x,y) = r(x)\cdot s(y)$$

lze zapsat ve tvaru

$$\int_{x_0}^{x(t)} \frac{ds}{g(s)} = \int_{t_0}^{t} f(s)\,ds \quad \forall t \in I.$$

$\boxed{\text{Důkaz:}}$

$$x'(t) = f(t)\, g(x(t)) \ \wedge \ g(x) \neq 0$$

$$\int_{t_0}^{t} \frac{x'(\tilde{t})}{g(x(\tilde{t}))}\,d\tilde{t} = \int_{t_0}^{t} f(\tilde{t})\,d\tilde{t} \quad \textcircled{=} \quad \left\{ \begin{array}{l} x(t) = s \\ x'(t)\,dt = ds \end{array} \right\} \ \dots \qquad \begin{array}{l} t_0 \to x(t_0) = x_0 \\ t \to x(t) \end{array}$$

$$\Rightarrow \int_{x(t_0)}^{x(t)} \frac{1}{g(s)}\,ds = \int_{t_0}^{t} f(s)\,ds \qquad \blacksquare$$

$\boxed{\text{Pozn.:}}$ $$\frac{dx}{dt} = f(t)\,g(x)$$

$$\frac{1}{g(x)}\,dx = f(t)\,dt$$ .... lze chápat jako rovnost diferenciálů.

$\boxed{\text{Př}}$

$$x\,y(x)\,y'(x) = 1 - x^2$$

$$y\,y' = \frac{1-x^2}{x}$$

$$\int y\,dy = \int \frac{1-x^2}{x}\,dx \quad \Rightarrow \quad \boxed{\frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + C}$$

$$y(x) = \pm\sqrt{2\ln|x| - x^2 + C_2}$$

PŘ

$$y' \cdot \operatorname{tg} x - y = a \qquad y = y(x)$$

$$\frac{dy}{dx} \operatorname{tg} x = y' \operatorname{tg} x = a + y$$

$$\int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x}\, dx$$

$\begin{cases} x = y \\ a^x = a^y \\ \text{- prostá funkce} \end{cases}$

$$\ln|a+y| = \ln|\sin x| + c$$

$$e^{\ln|a+y|} = e^{\ln|\sin x| + c} = k \cdot e^{\ln(\sin x)} \quad \{ k = e^c$$

$$|a+y| = k \cdot \sin x$$

$$\boxed{y = k \cdot \sin x - a}$$

PŘ

$$(x y^2 + x)\,dx + (y - x^2 y)\,dy = 0$$

$$x(y^2+1)\,dx + y(1-x^2)\,dy = 0$$

$$\frac{1}{2}\int \frac{2y}{y^2+1}\,dy = -\frac{1}{2}\int \frac{2x}{1-x^2}\,dx$$

$$\frac{1}{2} \ln|y^2+1| = +\frac{1}{2} \ln|1-x^2| + C \quad / \cdot 2$$

$$\cancel{e^{\frac{1}{2}\ln(y^2+1)}}$$

$$e^{\ln(y^2+1)} = e^{\ln(1-x^2) + \tilde{C}} = e^{\ln(1-x^2)} \cdot K$$

$$\boxed{y^2 + 1 = |1-x^2| \cdot K}$$

$$y(0) = 2 \Rightarrow 5 = 1 \cdot k \Rightarrow k = 5$$

ROVNICE PŘEVODITELNÉ NA RCE (S)

• ROVNICE "S PŘÍMKOU"

$$y'(x) = f(ax+by+c)$$

→ použijeme substituce $u = ax+by+c$ a dostaneme rci (S)

PŘ

$$y' = \sin(x-y)$$

$$\boxed{u = x-y}$$

$$u'(x) = 1 - y'(x) \Rightarrow y' = 1-u'$$

$$1-u' = \sin(u)$$

$$\frac{du}{dx} = u' = 1-\sin(u)$$

$$\int \frac{du}{1-\sin u} = \int dx = x + C$$

$$\int \frac{du}{1-\sin u} = \int \frac{1+\sin u}{1-\sin^2 u}\,du = \int \frac{1+\sin u}{\cos^2 u}\,du = \operatorname{tg} u + \int \frac{\sin u}{\cos^2 u} = \operatorname{tg} u + \frac{1}{\cos u}$$

$$\operatorname{tg} u + \frac{1}{\cos u} = x + C \Rightarrow \operatorname{tg}(x-y) + \frac{1}{\cos(x-y)} - x = C$$

ZKOUŠKA: (derivace)

$$\frac{1}{\cos^2(x-y)}(1-y') - \frac{-\sin(x-y)}{\cos^2(x-y)}(1-y') - 1 = 0$$

$$[1-\sin(x-y)]\cdot\left[\frac{1}{\cos^2(x-y)}+\frac{\sin(x-y)}{\cos^2(x-y)}\right]-1$$

$$=\cancel{\frac{1-\sin(x-y)-\cos^2(x-y)}{\cos^2(x-y)}}$$

$$\frac{1-\sin^2(x-y)}{\cos^2(x-y)}-1=0. \checkmark$$

$$y'=1+\sin(x-y)$$

$$\vdots$$

$$\Rightarrow \left[\ln\left(\operatorname{tg}\frac{x-y}{2}\right)=-x+C\right]$$