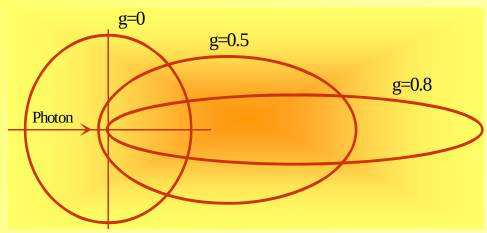
http://en.wikipedia.org/wiki/Boltzmann_equation

The Transport Equation



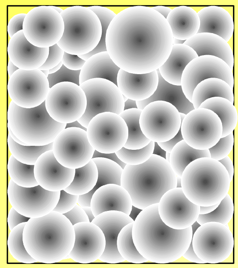
Medical Optics

Stefan Andersson-Engels

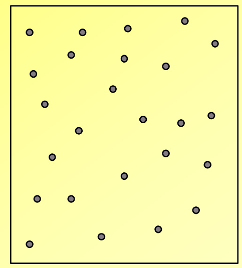
Dualism of light

- Wave picture: light is considered as an electromagnetic wave modelled by the Maxwell equations
- Particle picture: light is considered as a stream of energetic particles - photons modelled by energy conservation - the transport equation

Discrete scatterers vs. continuous variation of n



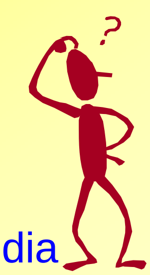
Continuous variation of n



Discrete scatterers

Transport equation - examples of applications

- heat conduction
- diffusion
- neutrons in nuclear reactions
- light propagation in turbid media



Transport equation

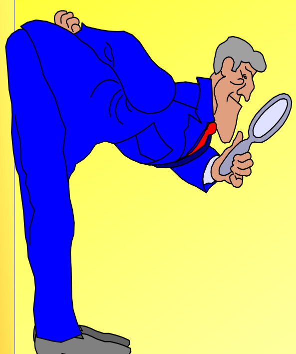
- Can be solved **analytically**
- Can be solved **numerically**

} ~ Simplifications are necessary

- Can be used to **simulate** the transport

– Monte Carlo simulations

Monte Carlo simulations



- No simplifications necessary
- Provides results for a specific case only - no analytical expressions
- Photon statistics limit the signal-to-noise ratio
- Extensive computer capacity is often required

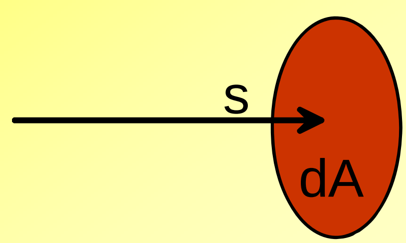
How do we reach an expression for light transport in tissue?

To get an expression that could be solved either analytically or numerically, we start to look into **conservation of energy** in a small volume of tissue

Perpetuum mobile ??

Radiance L(r,s,t)

Radiance L(**r**,**s**,t) [W/m^{2}sr] is the quantity used to describe the propagation of photon power. It is defined as the radiant power per unit solid angle about the unit vector **s** and per unit area perpendicular to **s**.



The photon distribution function N(r,s,t)

N(**r**,**s**,t)d³rdω is the number of photons in the volume d³r with the direction **s** within dω at time t.

The unit of N(**r**,**s**,t) is photons $m^{-3}sr^{-1}$.

The radiance or intensity is obtained by multiplying N by the photon energy and the velocity of light:

$$L(\mathbf{r},\mathbf{s},t) = N(\mathbf{r},\mathbf{s},t)h\nu c$$

Radiant Energy Fluence Rate φ(r,t)

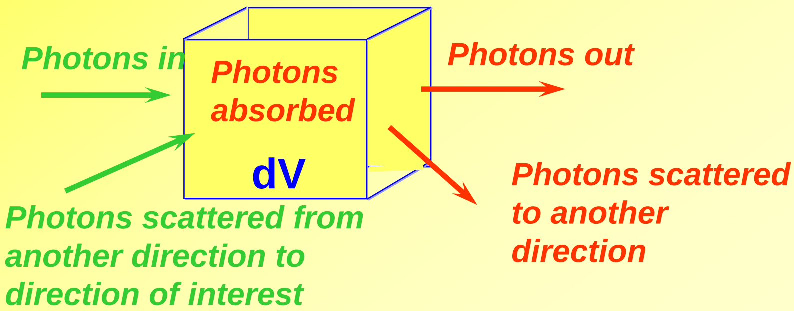
Absorption of photons is the key to most clinical applications. Because an absorption chromophore located at **r** can absorb photons **irrespectively of their direction of propagation**, the integral of the radiance over all directions, called the fluence rate ϕ(r) [W/m²], has more practical significance than radiance itself.

$$\phi(\mathbf{r},t) = \int_{4\pi} L(\mathbf{r},\mathbf{s},t)d\omega = ch\nu\int_{4\pi} N(\mathbf{r},\mathbf{s},t)d\omega$$

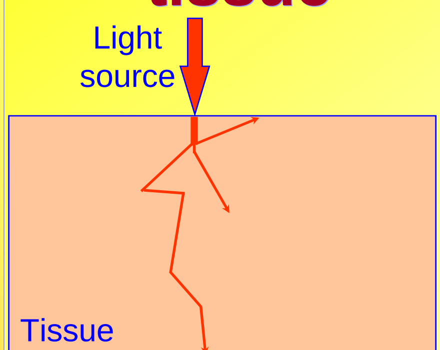
The fluence rate is defined as the radiant power incident on a small sphere, divided by the cross sectional area of that sphere

The transport equation (I)

- Assume a small volume dV and a direction s.
- Conservation of energy yields that photons can only be added or subtracted from the photon distribution function in specific interactions.



Light transport in tissue



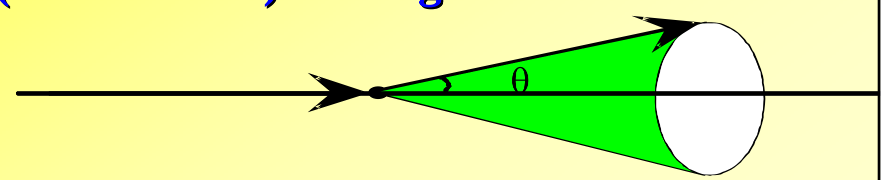
- Absorption, μ_a [m^{-1}]
- Scattering, μ_s [m^{-1}]
- Scattering phase function

$\mu_s >> \mu_a$ ➡ Diffusion

Scattering phase function

p(**s,s´)** is the probability function for a scattering from direction **s** to direction **s´**

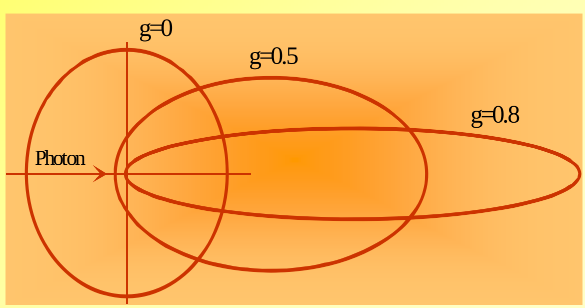
It is generally assumed that the scattering probability depends only on (the cosine of) the angle between s and s´:



The g-factor

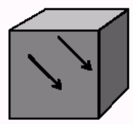
The *g*-factor is defined as $g = <\cos\theta>$

This is the parameter usually used in tissue optics to describe the anglular distribution of the light scattering



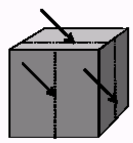
The terms in the transport equation

1) The change of photon distribution function



$$\int_V \frac{\partial N(\mathbf{r},\mathbf{s},t)}{\partial t} dV$$

2) Photons lost through boundary surface

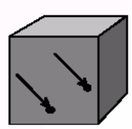


$$-\oint_S cN(\mathbf{r},\mathbf{s},t)\mathbf{s}\cdot d\mathbf{S}$$

$$= -\int_V c\mathbf{s}\cdot\nabla N(\mathbf{r},\mathbf{s},t)dV$$

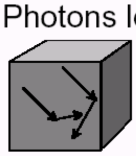
The terms in the transport equation (2)

3) Photons lost through absorption



$$-\int_V c\mu_a(\mathbf{r})N(\mathbf{r},\mathbf{s},t)dV$$

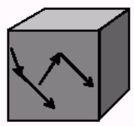
4) Photons lost through scattering



$$-\int_V c\mu_s(\mathbf{r})N(\mathbf{r},\mathbf{s},t)dV$$

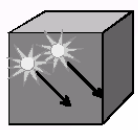
The terms in the transport equation (3)

5) Photons gained through scattering



$$+\int_V c\mu_s(\mathbf{r})\int_{4\pi} p(\mathbf{s}',\mathbf{s})N(\mathbf{r},\mathbf{s}',t)ds'dV$$

6) Plus a possible light source



$$+\int_V q(\mathbf{r},s,t)dV$$

Transport equation: Change

The time-resolved transport equation is a mathematical expression of the build-up of the photon density function N(**r**,**s**,t).

Thus the first term expresses changes in the photon distribution function with time:

$$\int_V \frac{\partial N(\mathbf{r},\mathbf{s},t)}{\partial t}dV$$

Transport equation: Crossing the border

Loss (I): Photons lost through the boundary. It can be expressed as a surface integral

$$-\oint_S cN(\mathbf{r},\mathbf{s},t)\mathbf{s}\cdot d\mathbf{S}$$

This can be re-expressed using Gauss´ theorem

$$= -\int_V c\mathbf{s}\cdot\nabla N(\mathbf{r},\mathbf{s},t)dV$$

Transport equation: Scattering

Loss (II): Photons scattered from direction **s** to any another direction **s**’

$$-\int_V c\mu_s(\mathbf{r})N(\mathbf{r},\mathbf{s},t)dV$$

Transport equation: Absorption

Loss (III): Photons incoming in direction **s** are absorbed

$$-\int_V c\mu_a(\mathbf{r})N(\mathbf{r},\mathbf{s},t)dV$$

Transport equation: Scattering into s

Gain (I): Photons gained through scattering from any direction **s**’ into the direction **s**

$$+\int_V c\mu_s(\mathbf{r})\int_{4\pi} p(\mathbf{s}',\mathbf{s})N(\mathbf{r},\mathbf{s}',t)ds'dV$$

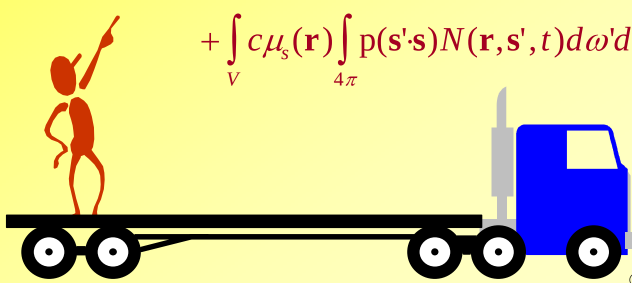
Transport equation: Sources

Gain (II): Photons gained through a light source q

$$+\int_V q(\mathbf{r},s,t)\,dV$$

The transport equation (II)

$$\int_V \frac{\partial N(\mathbf{r},\mathbf{s},t)}{\partial t}dV = -\int_V c\mathbf{s}\cdot\nabla N(\mathbf{r},\mathbf{s},t)dV$$
$$-\int_V c\mu_s(\mathbf{r})N(\mathbf{r},\mathbf{s},t)dV - \int_V c\mu_a(\mathbf{r})N(\mathbf{r},\mathbf{s},t)dV$$
$$+\int_V c\mu_s(\mathbf{r})\int_{4\pi} \mathrm{p}(\mathbf{s}'\cdot\mathbf{s})N(\mathbf{r},\mathbf{s}',t)d\omega' dV + \int_V \mathrm{q}(\mathbf{r},\mathbf{s},t)dV$$



Energy conservation

Note:

- The transport equation is derived under the assumption that the energy of the non-absorbed photons are kept the same despite all interactions with the medium, i.e. no loss or gain in photon energy due to the interaction.
- We restrict further the analysis to monochromatic light

Transport equation (III)

The transport equation is usually presented for the radiance L(**r**,**s**,t) [W m^{-2} sr^{-1}], and after dropping the integrals

$$\frac{1}{c}\frac{\partial L(\mathbf{r},\mathbf{s},t)}{\partial t}+\mathbf{s}\cdot\nabla L(\mathbf{r},\mathbf{s},t)+(\mu_s+\mu_a)L(\mathbf{r},\mathbf{s},t)=$$

$$=\mu_s\iint_{4\pi}L(\mathbf{r},\mathbf{s}',t)\,p(\mathbf{s},\mathbf{s}')\,d\omega'+Q(\mathbf{r},\mathbf{s},t)$$

Transport equation (IIIb)

The time-independent transport equation, valid for a steady-state situation, is derived by using that the radiance is independent of time, and thus is the time derivative of the radiance equal to zero

$$\mathbf{s}\cdot\nabla L(\mathbf{r},\mathbf{s}) = \frac{\partial L(\mathbf{r},\mathbf{s})}{\partial s} = -(\mu_s + \mu_a) L(\mathbf{r},\mathbf{s}) +$$

$$+ \mu_s \iint_{4\pi} L(\mathbf{r},\mathbf{s}') p(\mathbf{s},\mathbf{s}') d\omega' + Q(\mathbf{r},\mathbf{s})$$

The scattering coefficient

The origin of the light scattering in tissue is difficult to describe. Tissue consists of a large number of randomly distributed scattering kernels. They may consist of small regions with a refractive index differing from the surrounding, as for example mitochondria, cell membranes and even whole cells. The main scattering structures in tissue are not yet fully described. The quantity used to describe the scattering is called *the scattering coefficient*, μ_s.

The unit of the scattering and absorption coefficients is 1/length, typically [mm^{-1}].

The absorption and scattering coefficients

Definitions:

- The probability of absorption in the infinitesimal distance *ds* is $\mu_a ds$
- The probability of scattering in the infinitesimal distance *ds* is $\mu_s ds$

The phase function of single scattering p(s,s´)

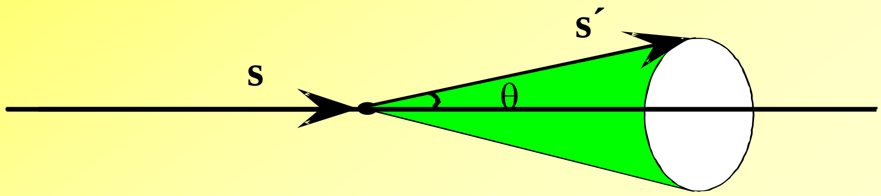
*p(***s,s´***)* is the probability function for a scattering from direction **s** to direction **s**´

$$\int_{4\pi} p(\mathbf{s},\mathbf{s}')d\omega' = 1$$

The phase function of single scattering p(s,s´)

It is generally assumed that the scattering probability depends only on (the cosine of) the angle between **s´** and **s**:

$$p(\mathbf{s},\mathbf{s}') = p(\mathbf{s}\cdot\mathbf{s}')$$



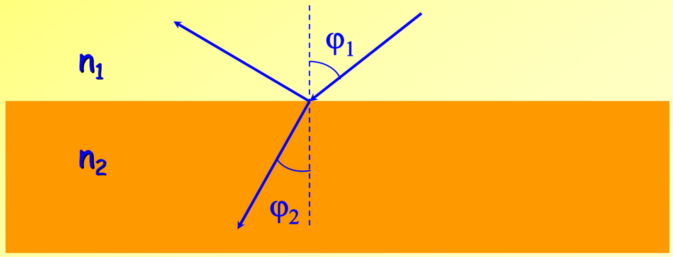
The scattering phase function

When light is scattered in tissue, the scattering angle at each scattering event has a probability distribution. This distribution is often referred to as the scattering phase function. A distribution that is often used in tissue optics is the **Henyey-Greenstein phase function**. This function was used in the original paper to describe the scattering of light in space by interstellar dust.

Reflection and Transmission at a surface

$$n_1 \sin\varphi_1 = n_2 \sin\varphi_2 \qquad \text{(Snell's law)}$$

$$r = \frac{1}{2}\left[\frac{\sin^2(\varphi_1 - \varphi_2)}{\sin^2(\varphi_1 + \varphi_2)} + \frac{\tan^2(\varphi_1 - \varphi_2)}{\tan^2(\varphi_1 + \varphi_2)}\right] \qquad \text{(Fresnel's law)}$$



The total attenuation coefficient and albedo

The total attenuation coefficient is defined as:

$$\mu_t = \mu_a + \mu_s$$

The ratio between the scattering coefficient and the total attenuation coefficient is called the albedo, a.

$$a = \frac{\mu_s}{\mu_t}$$

Albedo $a = \dfrac{\mu_s}{\mu_a + \mu_s}$

***Albedo**: whiteness, from Latin albus white*
***1)** reflective power **2) a:** the fraction of incident light or electromagnetic radiation that is reflected by a surface or body (as the moon, a planet, a cloud or a field of snow) **b:** the fraction of incident neutrons that have been reflected by a surface **3)** the whitish inner portion of the rind of citrus fruits*

Websters third new international dictionary, 1986

Radiance L(r,s,t)

Radiance L(**r**,**s**,t) [W/m^{2}sr] is the quantity used to describe the propagation of photon power. It is defined as the radiant power per unit solid angle about the unit vector **s** and per unit area perpendicular to **s**.

Radiance L(**r**,**s**,t) is related to the power dP(**r**,**s**,t) [W] flowing through infinitesimal area dA, located at **r**, in the direction of unit vector **s** (with an angle θ between **s** and the normal **n** of dA), within the solid angle ω:

$$dP(\mathbf{r},\mathbf{s},t) = L(\mathbf{r},\mathbf{s},t)\, dA \cos\theta\, d\omega$$